XIV RiNEM
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Special Opening Session: Ray Chaos, Full Wave Oddities and Regular to Irregular Transition in EM Scattering from Complex Systems

Bouncing Ray Chaos

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The WavesGroup
http://www.thewavesgroup.unisannio.it

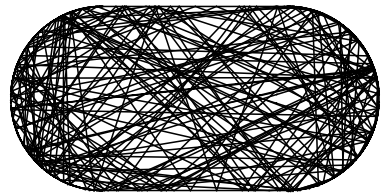
Agenda

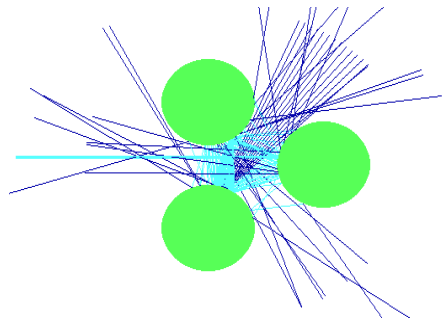
- Ray-chaotic boundary-value problems (BVPs);
- Heuristics: From Fermi's bouncing balls to bouncing photons/rays;
- A New paradigm in EM ray-chaos
  - Both internal and external,
  - Ray-optical analysis,
  - Full-wave analysis (TE incidence, internal and external fields);
- Digression: "Wave chaology";
- Conclusions and open issues.

Ray-chaotic BVPs

- **Chaos** = *exponentially* sensitive dependence on initial conditions in *deterministic* systems
  ➡ *complex/unpredictable/irregular* behavior;

- **Ray** (point-particle) chaos paradigms:
  - The Sinai-Bunimovich (internal) billiards
    [Sinai, *Russ. Math.Surv.* **25**, 137, 1970];
  - The 3-disk (external) scattering problem
    [Blumel & Smilanski, *Phys. Rev.Lett.* **60**, 477, 1988].

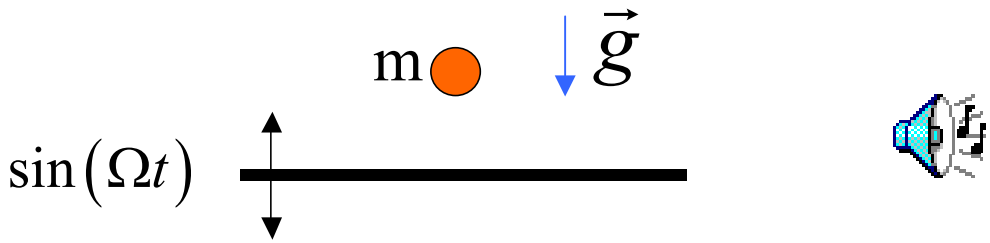




More *obvious* chaotic phenomena in EM systems show up as an effect of constitutive non-linearity, non-linear radiation pressure [Pinto & Pierro, *Phys. Lett.* **A185**, 14, 1994], etc.

Heuristics: From Fermi's bouncing balls to bouncing photons

- A pointlike ball falling under gravity upon an elastic vibrating table



- It was studied by Pustil'nikov [*Trans. Moscow Math. Soc.* **2**,1,1978] in connection with Fermi's model of charged particle acceleration by interaction with moving magnetic fields, relevant to cosmic rays [*Phys. Rev.* **75**, 1169, 1949].

- It became popular as a nicely viable illustration of KAM theorem in 2-degrees of freedom [see, e.g., Lichtenberg & Lieberman, *Regular and Stochastic Motion.,* Springer Verl., Berlin,1990].

Heuristics: From Fermi's bouncing balls to bouncing photons (cont'd)

- A similar behavior is observed by allowing a *nonzero* (albeit small) *initial transverse momentum* component, and a *fixed, periodic-in-space* (undulated) *wall* [Pierro & Pinto, 1999].

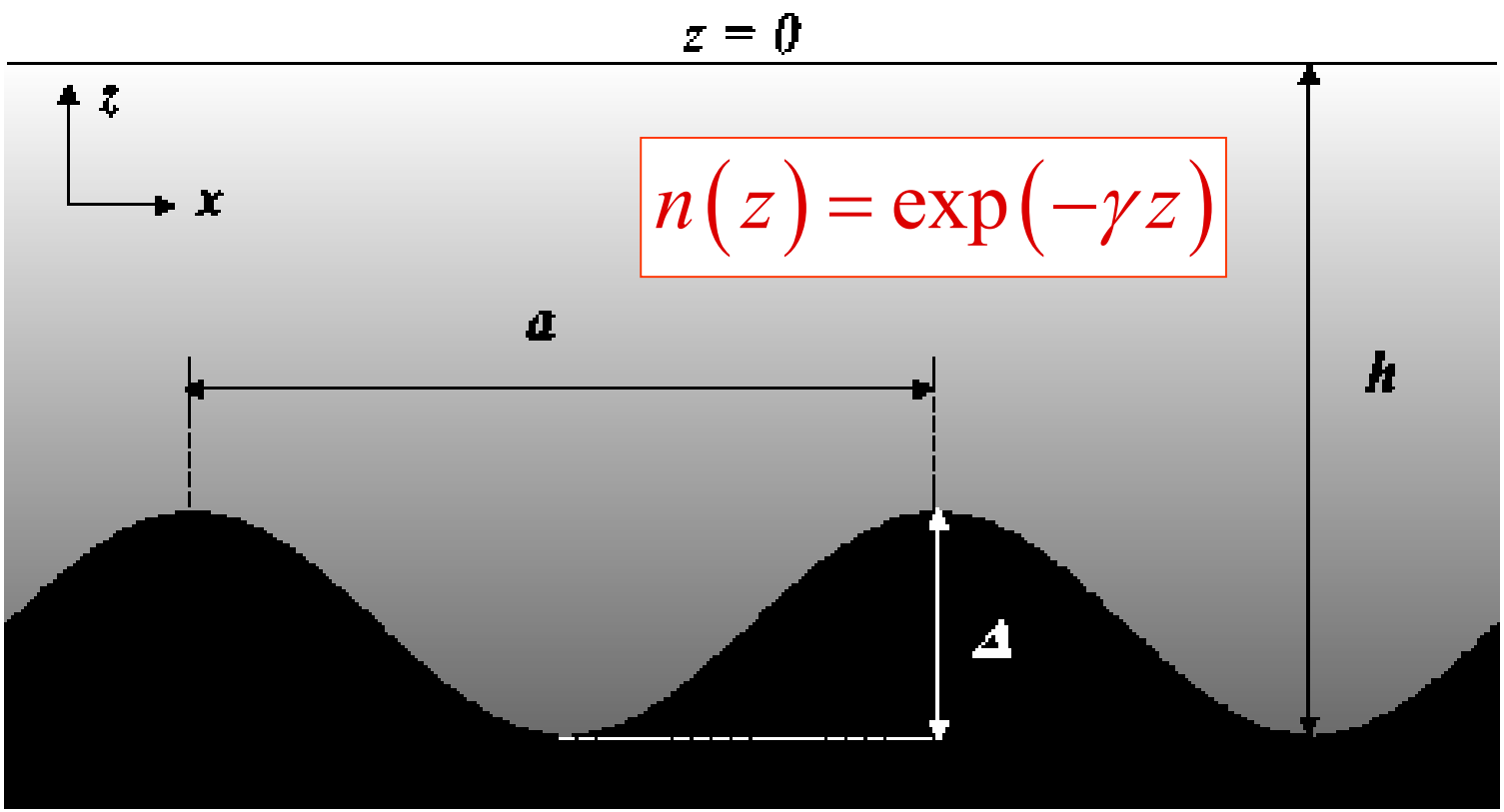
- The photon analog of the above *modified* Fermi-Pustyl'nikov problem is obtained by propagating an *oblique* ($k_x \neq 0$) ray (photon) through a *graded-index* dielectric slab:

$$n(\vec{r}) = n_0 e^{-|\gamma| z}$$

backed by an *undulated* (2-D) metallic surface, in the paraxial approximation.

Problem geometry

Undulated perfectly-conducting surface topped by a tapered (lossless) dielectric, vacuum-matched at $z=0$.



$$n(z) = \exp(-\gamma z)$$

Part – I

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Ray Analysis

Ray trajectories: Test case 1

/* Coating parameters */

h = 0.75; /* Coating thickness */

n(0) = 20.0 /* Coating refr. index at z=0 */

n(h) = 1.0 /* Coating refr. index at z=h */

/* Backing conductor profile */

$\zeta(x) = z_0 + a \log[c + b \cos(2\pi x/a)]$

/* Backing conductor profile parameters */

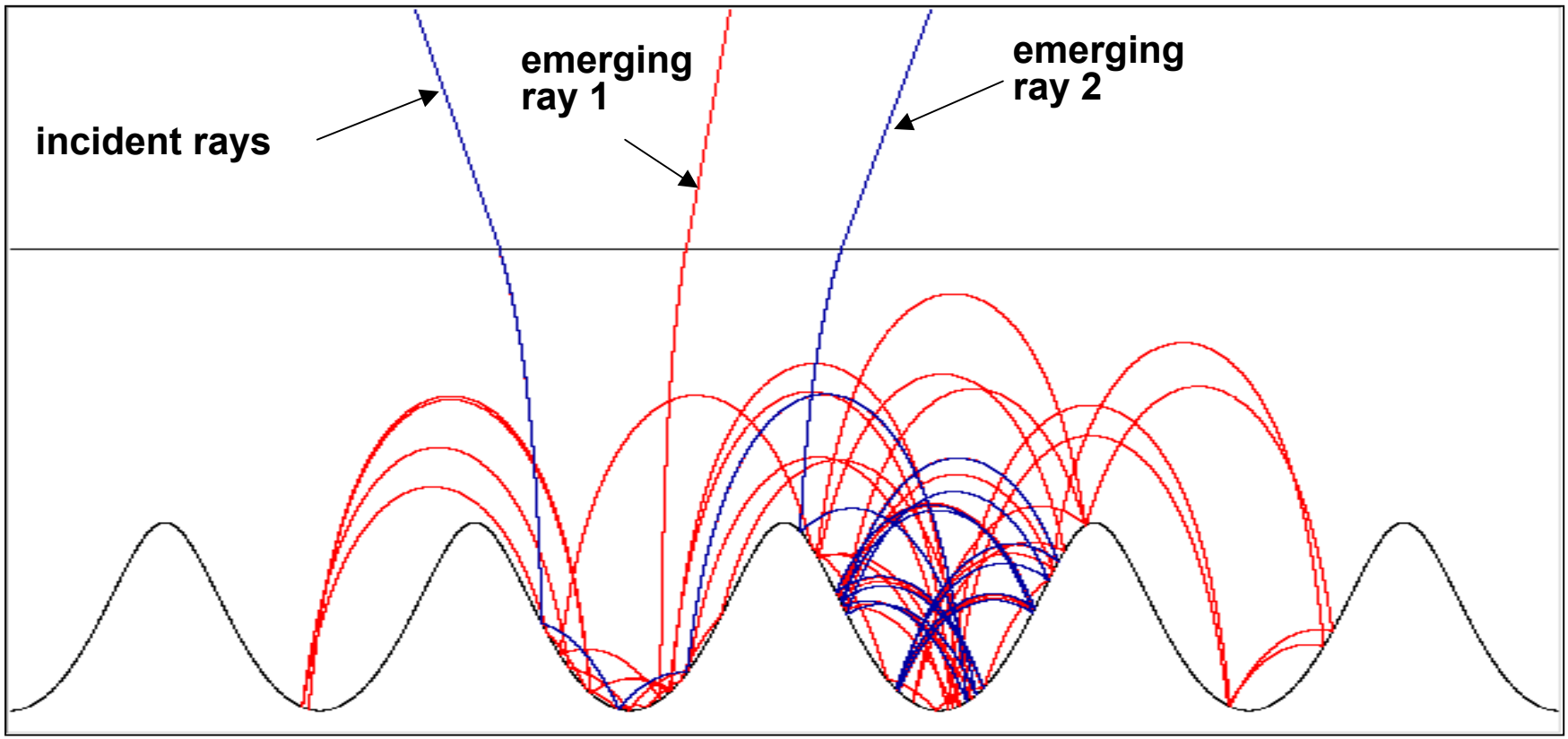
$b = 1.05;\ c = 2.05;\ a = 0.8;\ z_0 = -0.5 + h;$

$\theta_i = \pi/9$, $\Delta\theta_i/\theta_i = 10^{-3}$ /* Incidence angles */

$x_i = 1.58\ a$ /* Incidence point */

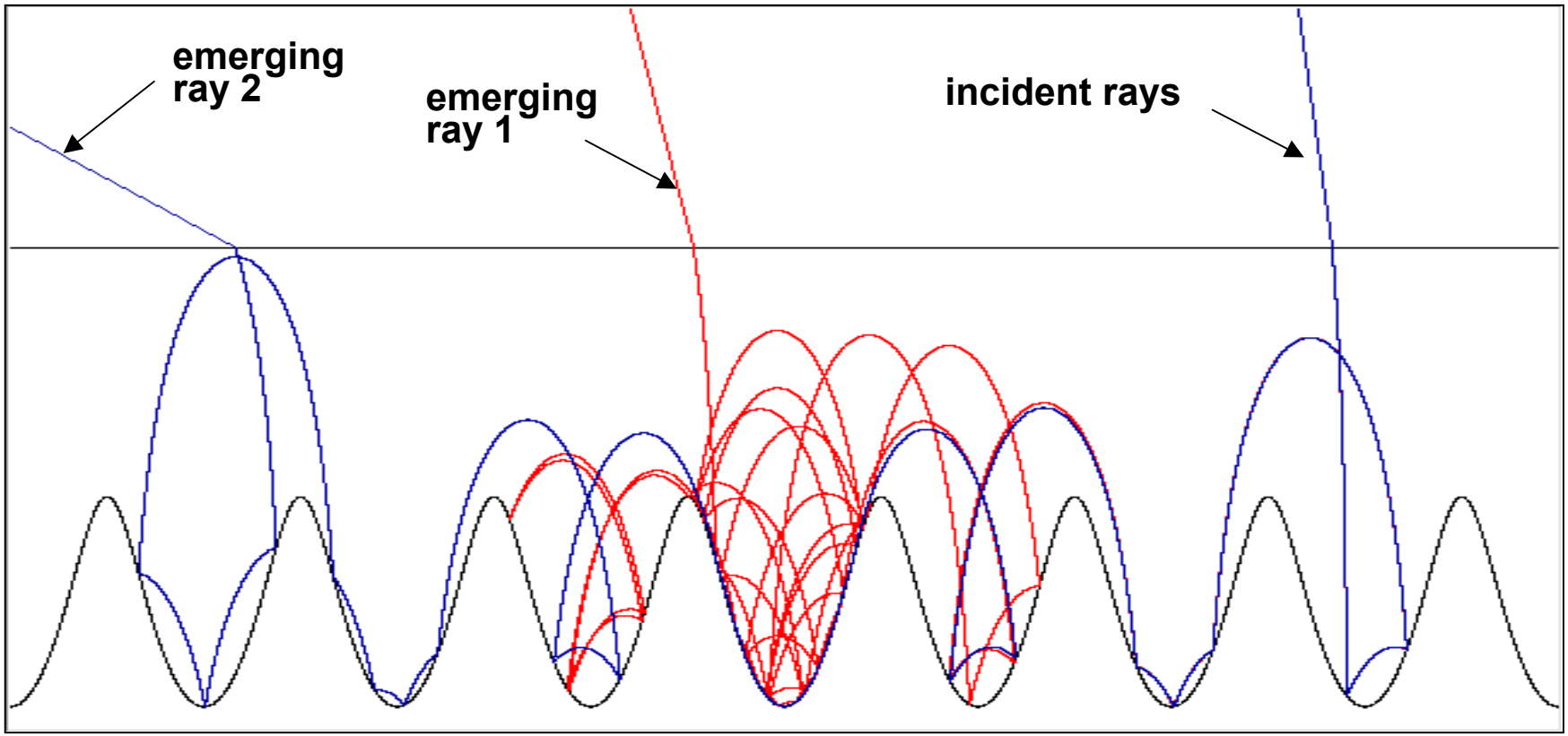
Ray trajectories

Nearly co-incident rays spread widely



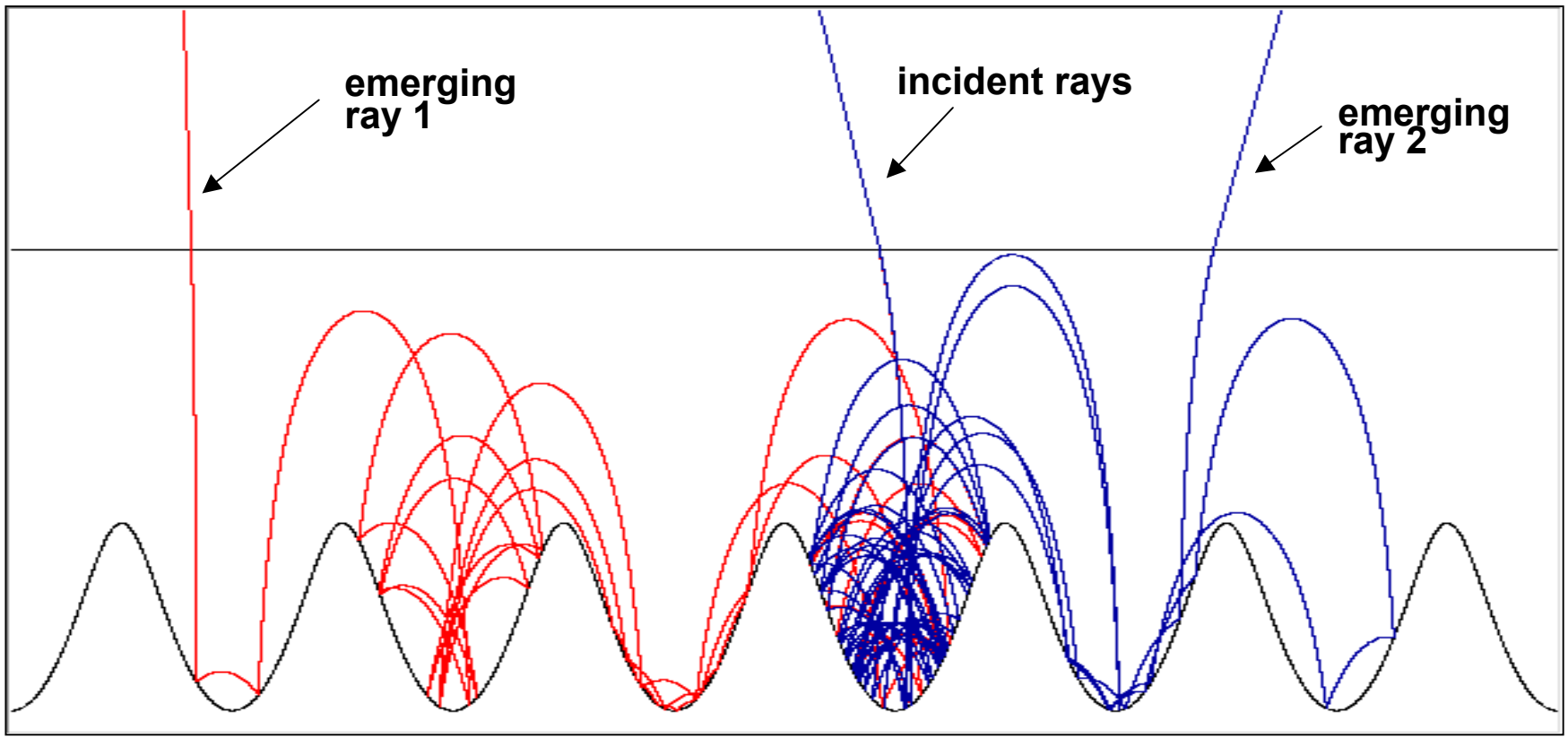
Ray trajectories (cont'd)

Nearly co-incident rays spread widely



Ray trajectories (cont'd)

Nearly co-incident rays spread widely



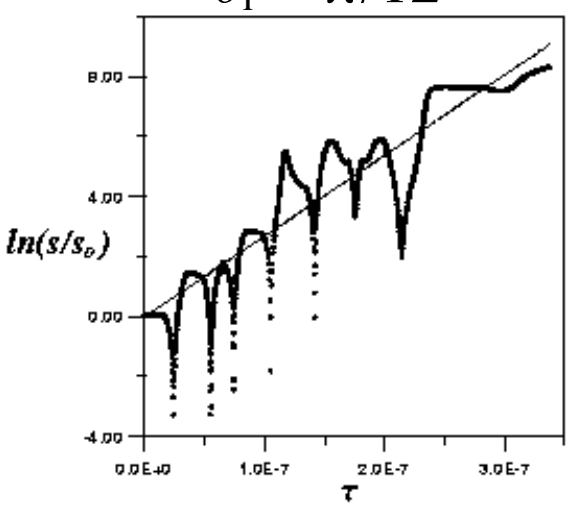
Ray trajectories (cont'd)

Ray paths split-up exponentially in time

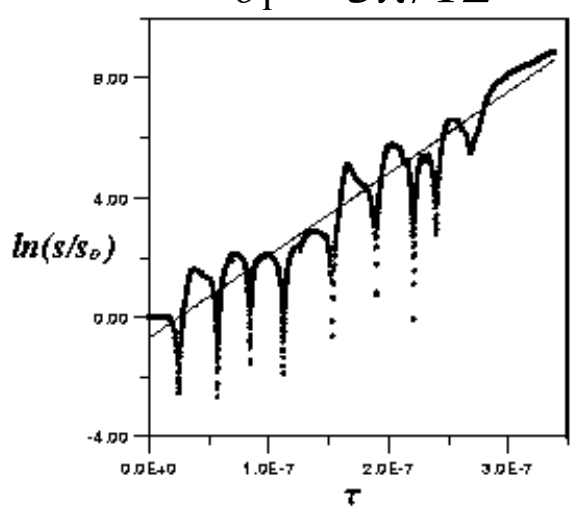
Examples:

- Averages over 100 different incidence points in $(0,a)$
- $n(0)=14$

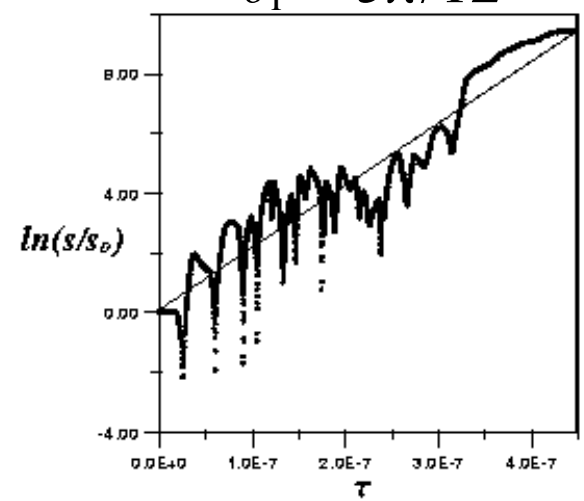
$\theta_i = \pi/12$



$\theta_i = 3\pi/12$



$\theta_i = 5\pi/12$



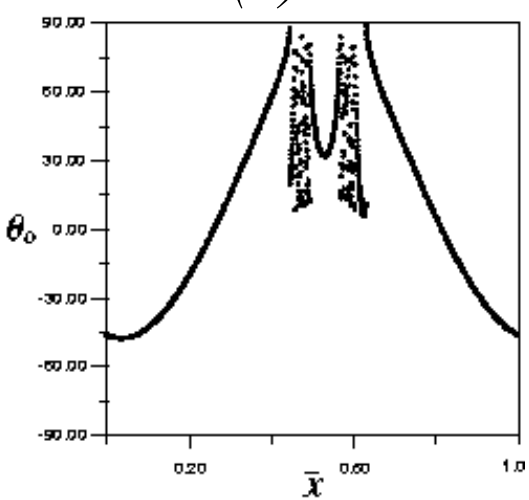
$$time = c^{-1}\int_{inc} n(s)ds$$

Ray trajectories (cont'd)

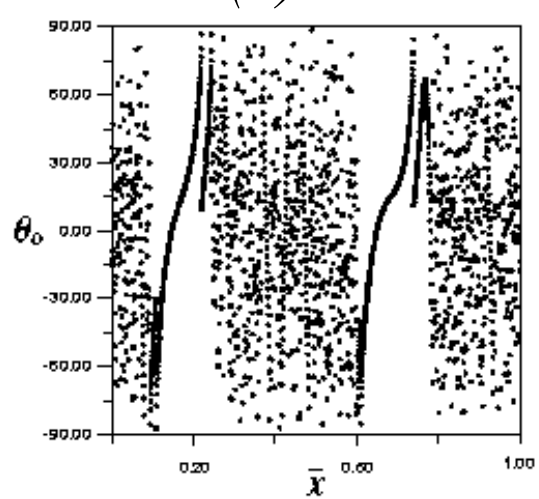
Measure of irregular subset of *(0,a)* blows up with *n(0)*

Examples: exit angle θ_o vs. scaled incidence abscissa $\bar{x} = x/a$ for $\theta_i = \pi/4$

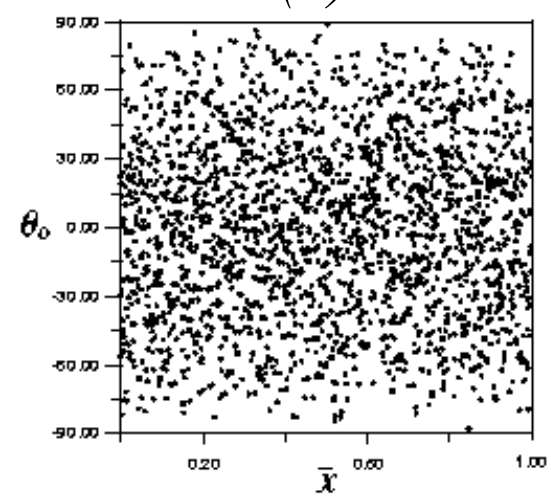
n(0)=1.4



n(0)=15



n(0)=90



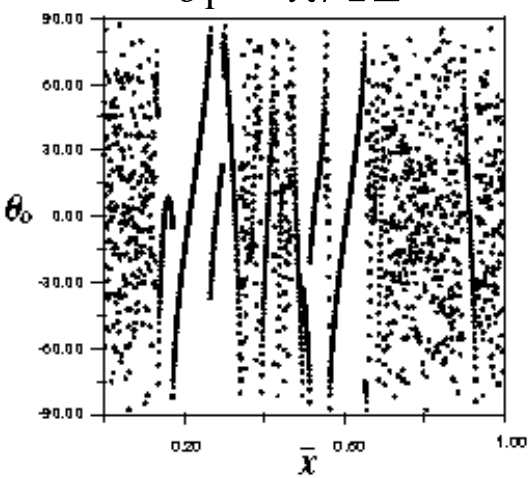
Ray trajectories (cont'd)

Measure of (ir)regular subset of *(0,a)* almost independent on incidence angle

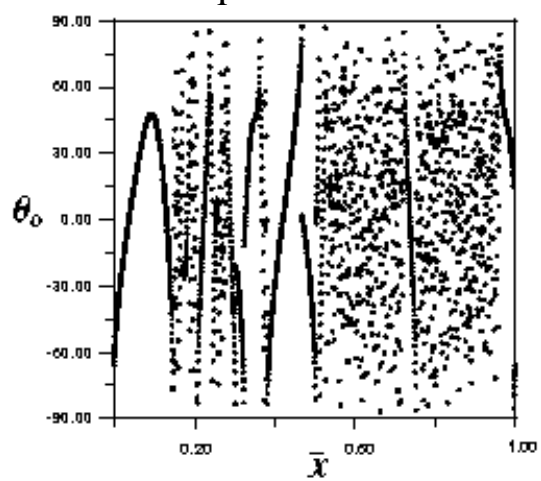
Examples: exit angle θ_o vs. scaled incidence abscissa

$\bar{x} = x/a$ for $n(0) = 14$

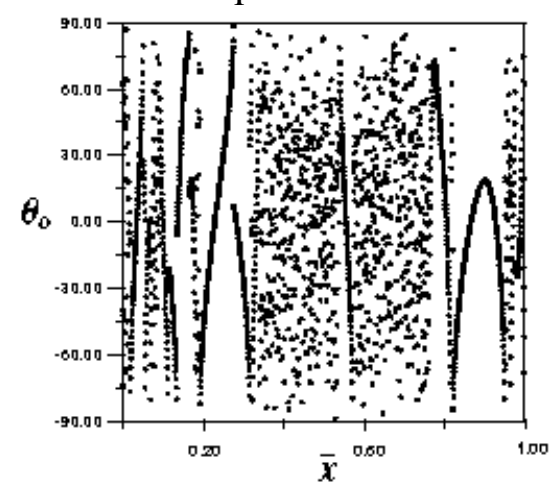
$\theta_i = \pi/12$



$\theta_i = 3\pi/12$



$\theta_i = 5\pi/12$



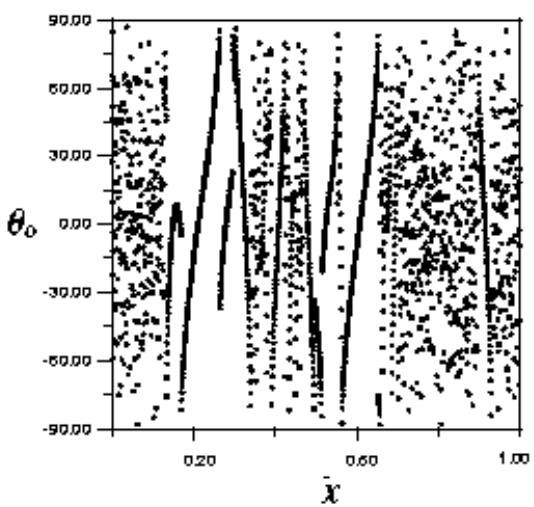
Ray trajectories (cont'd)

Regular islands observed at all scales

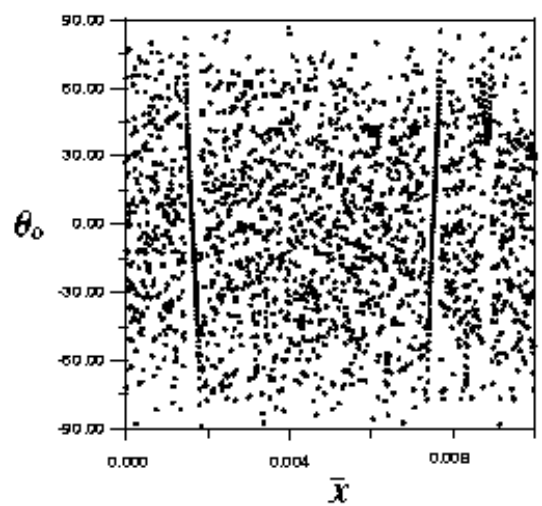
Examples: exit angle θ_o vs. scaled incidence abscissa

$\bar{x} = x/a$ for $n(0) = 15$ and $\theta_i = \pi/4$

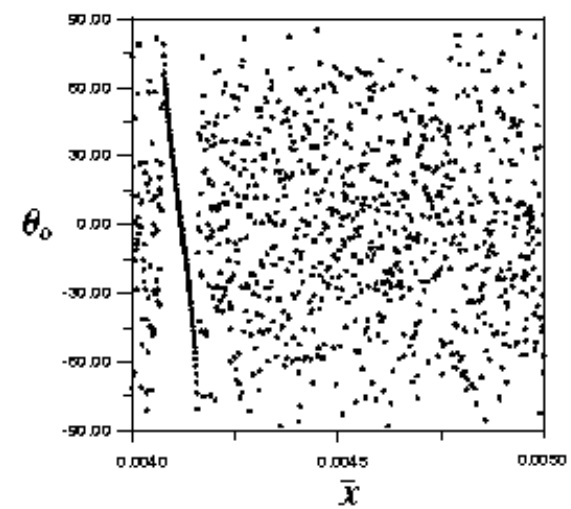
$\bar{x} \in (0,1)$



$\bar{x} \in (0, 0.01)$



$\bar{x} \in (.004, .005)$

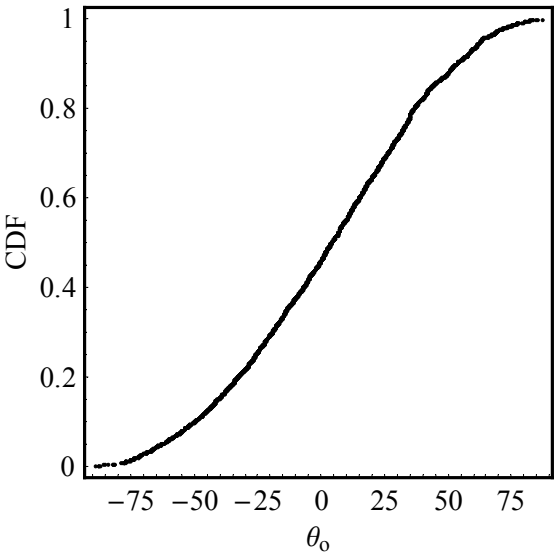


Ray trajectories (cont'd)

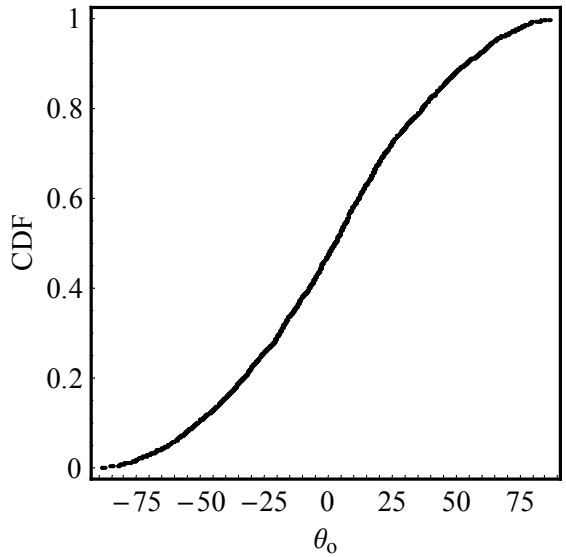
Cumulative distributions of exit angles

- 10^4 incident rays in a cone of $\Delta\theta \cong 10^{-3}$
- $n(0) = 14$

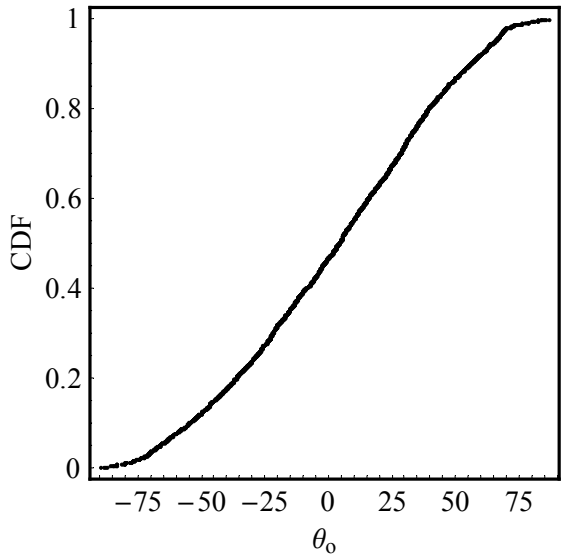
$\theta_i = \pi/12$



$\theta_i = 3\pi/12$



$\theta_i = 5\pi/12$



Exit angles almost uniformly distributed in [-90°,90°]

Ray trajectories: Concluding remarks

- Rays tend to be *trapped*, executing many back-forth hops before escaping;
- Nearby incident rays spread *exponentially* (chaos!);
- Narrow incident beams produce scattered fields with *uniform* exit-phase & exit-angle distribution at any incident angle
  - Potential applications: radar countermeasures;
- If losses are included, strong attenuation of exiting rays & uniform power deposition (no hot-spots) might be expected
  - Potential applications: smart absorbers.

Part – II

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Full-Wave Analysis

- Time-harmonic $\exp(j\omega t)$
- TE incidence

Full-wave solution - TE

- Assume exponential dielectric profile, vacuum-matched at *z=0*

$$\varepsilon(z) = \begin{cases} \varepsilon_0, & z \geq 0 \\ \varepsilon_0 \exp(\gamma z), & z < 0 \end{cases}$$

- Use Maxwell ($\mathrm{div}\,\mathbf{D}$) equation to show that the (only nonzero component of the) electric field is ruled by Helmholtz equation:

$$\nabla^2 E + k^2(z)E = 0$$

- Expand internal & reflected fields into appropriate Floquet series

$$\vec{E}^{(i)} = E_0 \hat{u}_y \exp\ -jk_0(x\sin\theta^i - z\cos\theta^i)$$

$$\vec{E}^{(t)} = E_0 \hat{u}_y \sum_{n=-\infty}^{\infty} c_n(k_0 z)\exp(-jk_{nx}x), \quad z < 0$$

$$\vec{E}^{(r)} = E_0 \hat{u}_y \sum_{n=-\infty}^{\infty} b_n \exp[-j(k_{nx}x + k_{nz}z)], \quad z \geq 0$$

Full-wave solution – TE (cont'd)

- Capitalize on known solution $u = C_\nu\left[\lambda e^z\right]$ of

$$\ddot{u} + \lambda^2 e^{2z} - \nu^2 \; u = 0$$

- ($\mathcal{C}_\nu$ any Bessel function) to write down internal expansion coefficients $c_n(z)$ as

$$c_n(z) = u_n U_n(k_0 z) + v_n V_n(k_0 z),$$

- where (Hankel)

$$(U,V)_n(k_0 z) = H^{(1,2)}_{\left|\frac{2k_{nx}}{\gamma}\right|}\left[\frac{2k_0}{|\gamma|}\exp\left(\frac{\gamma}{2k_0}k_0 z\right)\right]$$

Full-wave solution – TE (cont'd)

- From $(\mathrm{curl}\ \mathbf{E})$ Maxwell equation

$$H_x = \frac{1}{j\omega\mu_0}\partial_z E_y$$

derive incident, reflected and internal transverse magnetic fields

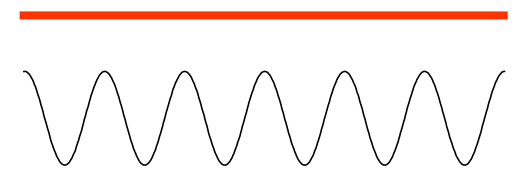
$$H_x^{(i)} = \frac{k_z}{\omega\mu_0} E_0 \exp\left\{j\left[k_x x + k_z z\right]\right\},$$

$$H_x^{(r)} = -\sum_{n=-\infty}^{\infty} \frac{k_{nz} b_n E_0}{\omega\mu_0} \exp\left\{j\left[n\frac{2\pi}{a} + k_x\right]x - jk_{nz}z\right\},$$

$$H_x^{(t)} = -\sum_{n=-\infty}^{\infty} \frac{\dot{c}_n(z)}{j\omega\mu_0} \exp\left\{j\left[n\frac{2\pi}{a} + k_x\right]x\right\}.$$

Full-wave solution – TE (cont'd)

- Use boundary conditions at $z=0$

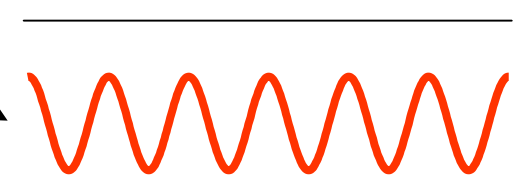


$$\begin{cases} \delta_{n0} + b_n = u_n U_n(0) + v_n V_n(0), \\ k_{zn}(\delta_{n0} - b_n) = -jk_0 \left[u_n \dot{U}_n(z) + v_n \dot{V}_n(z) \right]_{z=0}. \end{cases}$$

to eliminate b_n,

$$u_n \left[k_{zn} U_n(0) - jk_0 \dot{U}_n(0) \right] =$$
$$= -v_n \left[k_{zn} V_n(0) - jk_0 \dot{V}_n(0) \right] + 2k_{zn}\delta_{n0}.$$

- Enforce boundary conditions at $z=\zeta(x)$



$$\sum_{n=-\infty}^{\infty} \{u_n U_n[\zeta(x)] + v_n V_n[\zeta(x)]\} \exp \; jn\frac{2\pi}{a}x \; = 0.$$

Full-wave solution – TE (cont'd)

- Adopt *friendly* profile

$$\zeta(x) = z_0 + \frac{2}{\gamma}\log\left[1 + b\cos\left(\frac{2\pi}{a}x\right)\right]$$

- for which

$$(U,V)_n[k_0\zeta(x)] = H_\nu^{(1,2)}\left[c' + b'\cos\left(\frac{2\pi}{a}x\right)\right]$$

where

$$\nu = \left|\frac{2}{\gamma}k_{nx}\right|, \quad c' = \frac{2k_0}{|\gamma|}\exp\left(\frac{\gamma z_0}{2}\right), \quad b' = \frac{2k_0}{|\gamma|}\ b\ \exp\left(\frac{\gamma z_0}{2}\right)$$

- Fourier expand Hankel functions:

$$H_\nu^{(1,2)}\left[c' + b'\cos\left(\frac{2\pi x}{a}\right)\right] = \sum_{m=-\infty}^{\infty} H_{\nu,m}^{(1,2)}\exp\left(jm\frac{2\pi x}{a}\right)$$

Full-wave solution – TE (cont'd)

- To compute Fourier coefficients of Hankel functions, use

- i) Neumann addition formula

$$H_\nu^{(1,2)}\left[c' + b'\cos\left(\frac{2\pi}{a}x\right)\right] = \sum_{k=-\infty}^{\infty} H_{\nu-k}^{(1,2)}(c') \cdot J_k\left[b'\cos\left(\frac{2\pi}{a}x\right)\right]$$

ii) Fourier Bessel expansion

$$J_k\left[b'\cos\left(\frac{2\pi}{a}x\right)\right] = \sum_{m=-\infty}^{\infty} J_{k,m}\exp\left(jm\frac{2\pi x}{a}\right)$$

iii) well known integral [Prudnikov *et al.*, *Integrals and Series*, Gordon & Breach, NY, 1992, 2.12.27]

$$J_{k,m} := \frac{1}{a}\int_{-a/2}^{a/2} dx \exp\left(-jm\frac{2\pi x}{a}\right) J_k\left(b'\cos\frac{2\pi x}{a}\right) = \frac{1}{2}\left[1+(-)^{m+k}\right] J_{\frac{k-m}{2}}\left(\frac{b'}{2}\right) J_{\frac{k+m}{2}}\left(\frac{b'}{2}\right)$$

Full-wave solution – TE (cont'd)

- The resulting expansions are relatively handy
  - Rapidly- converging, easily computable

- Boundary conditions at $z=\zeta(x)$ become

$$\sum_{n=-\infty}^{+\infty}\left(u_n H^{(1)}_{\nu,p-n} + v_n H^{(2)}_{\nu,p-n}\right) = 0, \quad p = -\infty, \ldots, \infty$$

where

$$H^{(1,2)}_{\nu,m} = \sum_{k=-\infty}^{\infty} H^{(1,2)}_{\nu-k}(c') \cdot J_{k,m}.$$

Full-wave solution – TE (cont'd)

- The problem is reduced to (truncation and) solution of an infinite linear system in the unknowns u_n, v_n

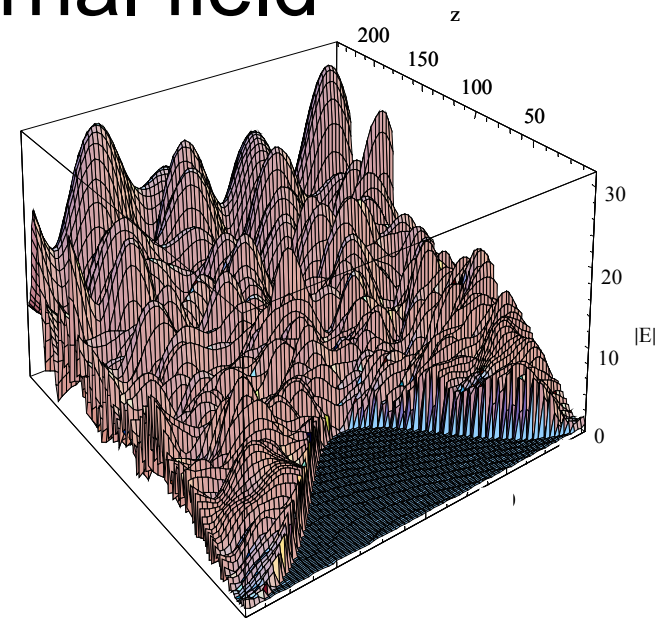
$$\begin{cases} \displaystyle\sum_{n=-\infty}^{+\infty} \left(u_n H^{(1)}_{\nu,p-n} + v_n H^{(2)}_{\nu,p-n} \right) = 0, \quad p = -\infty, \ldots, \infty \\ u_n \left[k_{zn} U_n(0) - j k_0 \dot{U}_n(0) \right] = \\ = \quad v_n \left[k_{zn} V_n(0) - j k_0 \dot{V}_n(0) \right] + 2 k_{zn} \delta_{n0}. \end{cases}$$

- As a rule-of-thumb , the system can be truncated at order $\sim a/\lambda(0)$

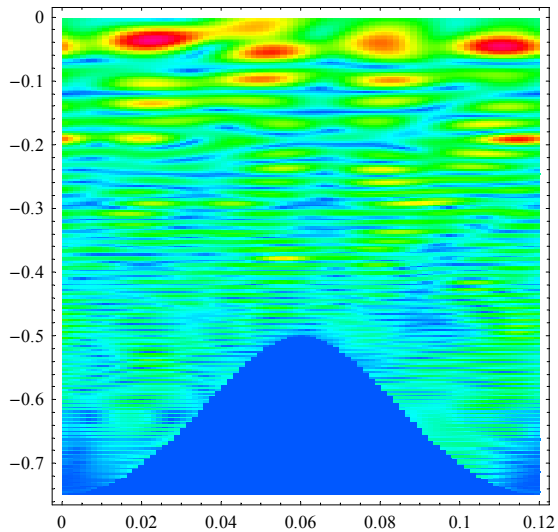
Full-wave simulations - Internal field

Parameters:
$a/\lambda=6,\ \theta_i=\pi/4$

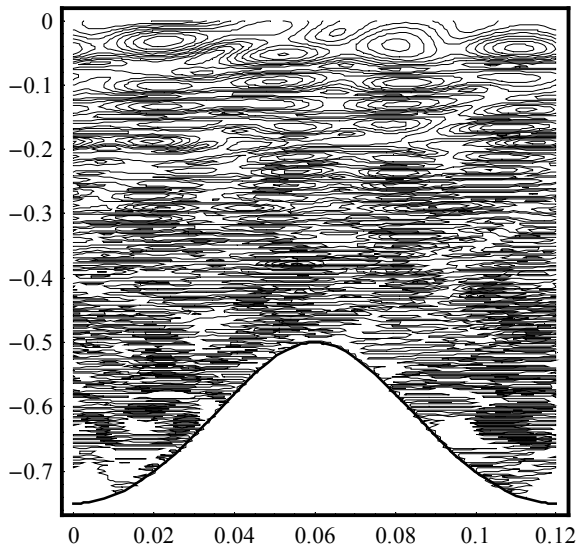
Intensity
(3D plot)



Intensity (levels)



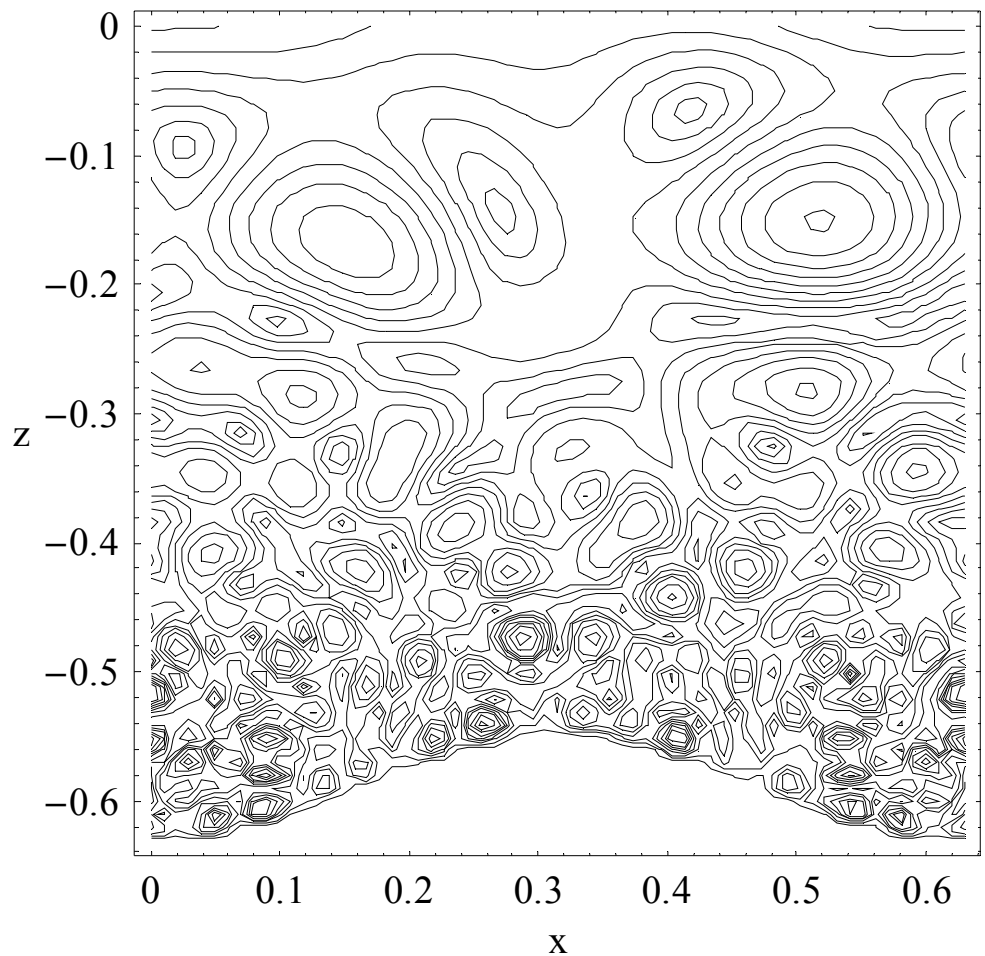
Intensity (contours)



Full-wave simulations - Internal field (cont'd)

Complex patterns set in at $a/\lambda \cong 1$

Nodal lines $(a/\lambda=1,\ \theta_i=\pi/4)$

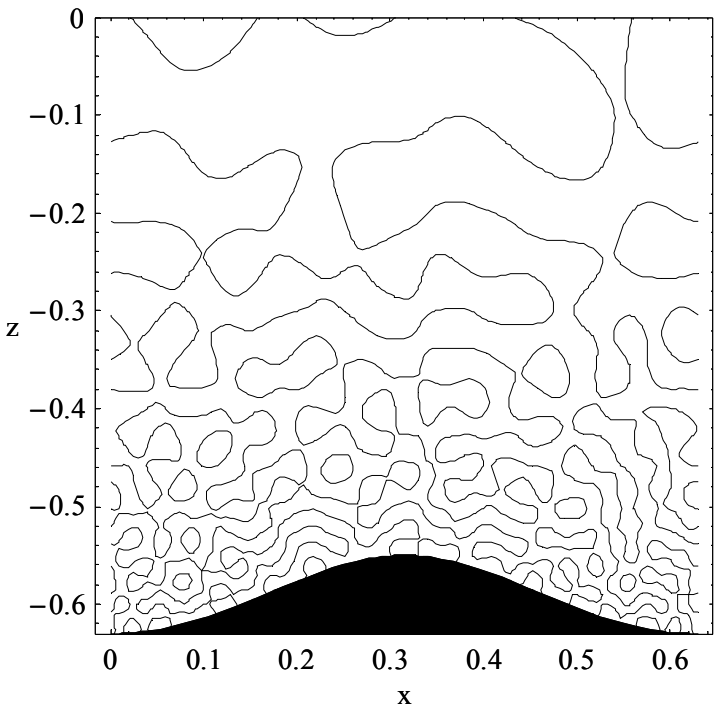


Full-wave simulations - Internal field (cont'd)

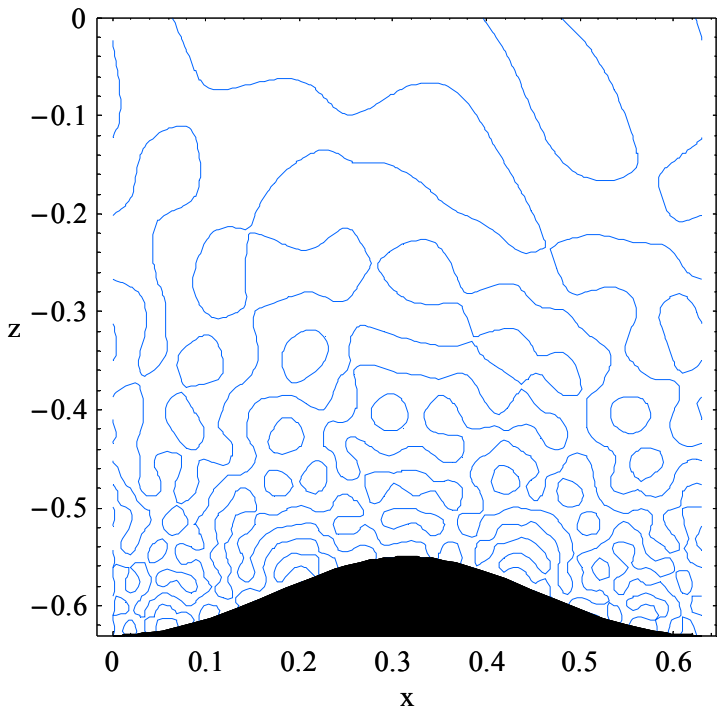
Meandric nodal lines (Chladni's phenomenon)

Example: $a/\lambda=2,\ \theta_i=\pi/4$

$Re[E]=0$



$Im[E]=0$



Digression: Wave chaology

- The general question is posed about the *fingerprints* (if any) of ray (asymptotic, short-wavelength) chaos in the full-wave regime

- Helmholtz equation (HE): $\nabla^2 E + k^2(z)E = 0$

  is the same as Schrödinger equation (SE) for point particle with mass *m* and energy *U* in a potential

$$k^2(z) = \frac{2m[U - V(z)]}{\hbar^2}$$

- Short wavelength stationary EM fields ruled by HE, correspond to semiclassical ($\hbar \to 0$) stationary solutions of SE,

- Classical chaos develops *complexity* on *arbitrary small* scales of phase space. Quantum mechanics implies *unobservability* of phase-space features *below* Planck's scale

- **Quantum (wave) chaos does *not* exist** [Berry, *Proc. Roy. Soc.*, **A413**, 183, 1987]

- **However, quantum (*wave*) systems which are chaotic in the semiclassical (*ray*) limit exhibit peculiar *features***

Digression: Wave chaology

Berry-Voros-Shnirelman conjecture

- The *averaged* (over a number of wavelengths) Wigner density *W* of a ray-chaotic field should correspond (in the limit as $\lambda \to 0$) to plane waves with random-uniform wave-vector directions through almost any point

$$\left\langle W(\vec{r},\vec{k})\right\rangle \approx \delta\left[k^2 - k^2(z)\right]$$

[Voros, *Ann. Inst. H. Poincare'* **24**, 31, 1976; Berry, *J. Phys.* **A10**, 2083, 1977]

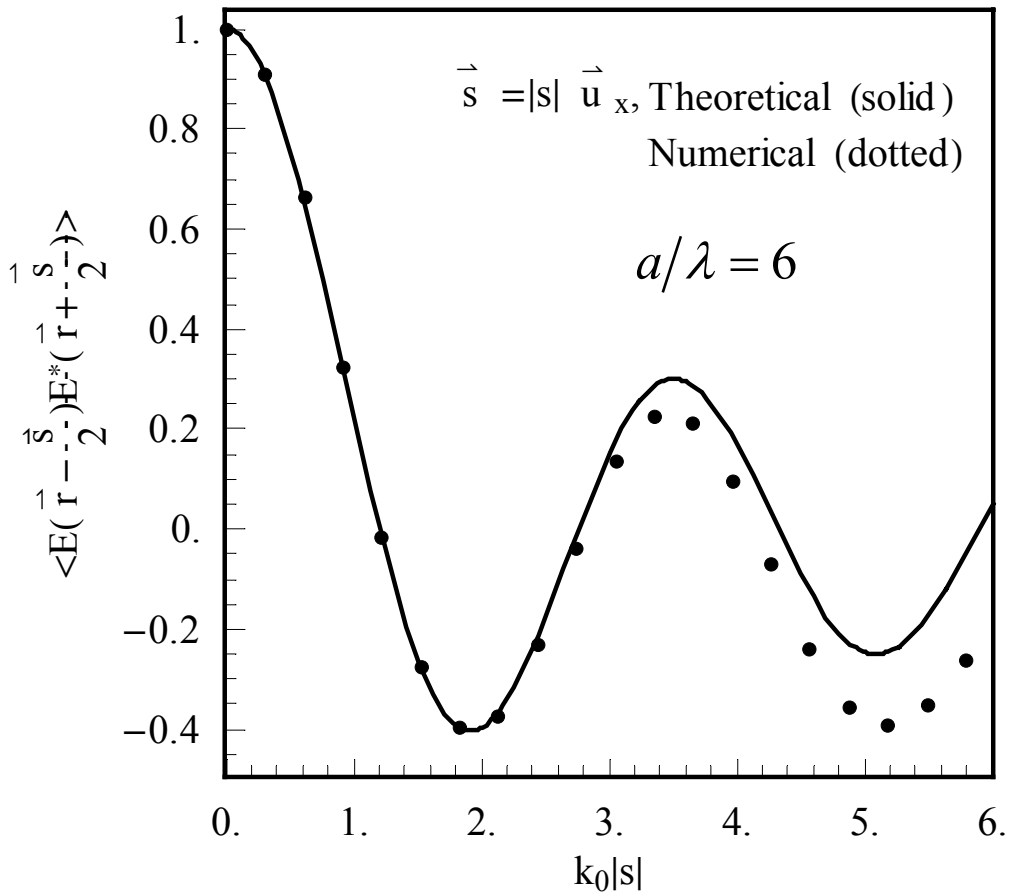
- As a result, one gets for the (smoothed) spatial field correlation

$$\overline{E\left(\vec{r}+\frac{\vec{s}}{2}\right)E^*\left(\vec{r}-\frac{\vec{s}}{2}\right)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_z \langle W(\vec{r},\vec{k})\rangle \exp(j\vec{k}\cdot\vec{s})$$

$$\propto k(z) J_0[k(z)|s|]$$

Full-wave simulations: correlation

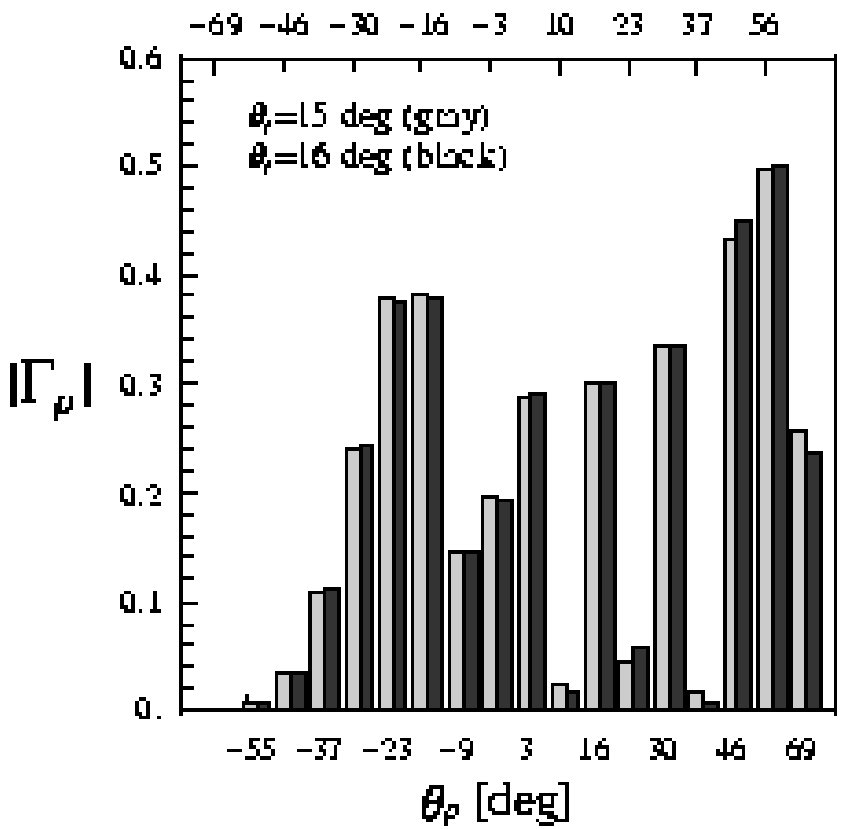
Internal field (smoothed) spatial correlation complies with Berry's conjecture

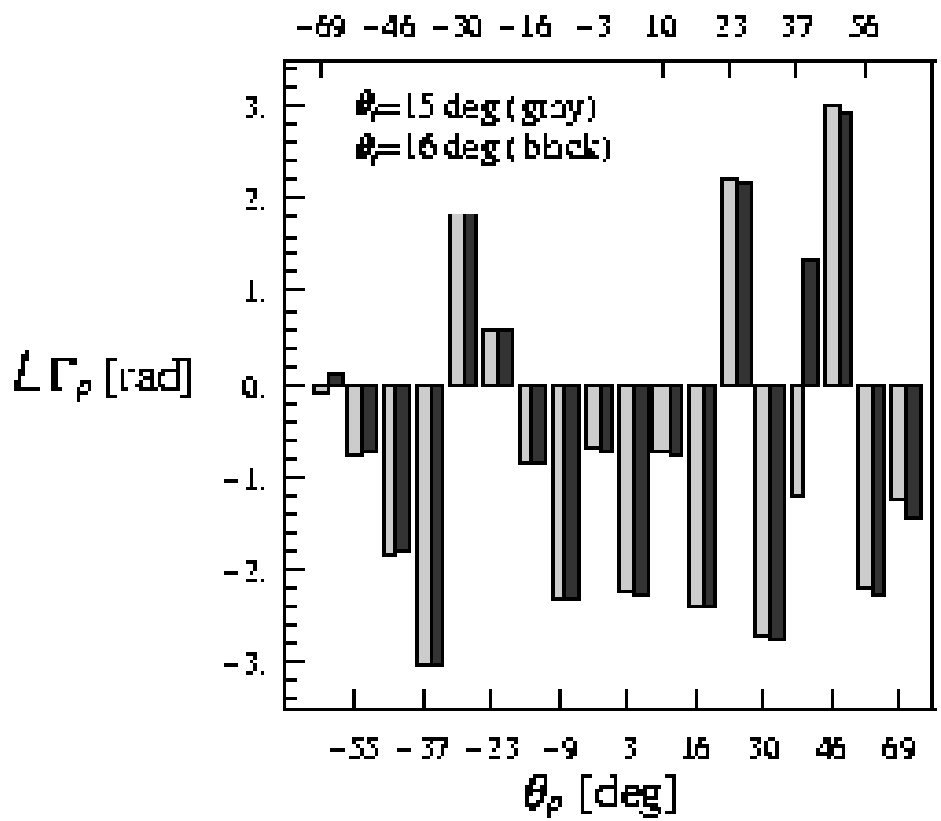


Full-wave simulations – Reflected field

Plain undulated surface (no dielectric topping) with $a/\lambda=9$

•Floquet coefficients depend smoothly on incidence angle.



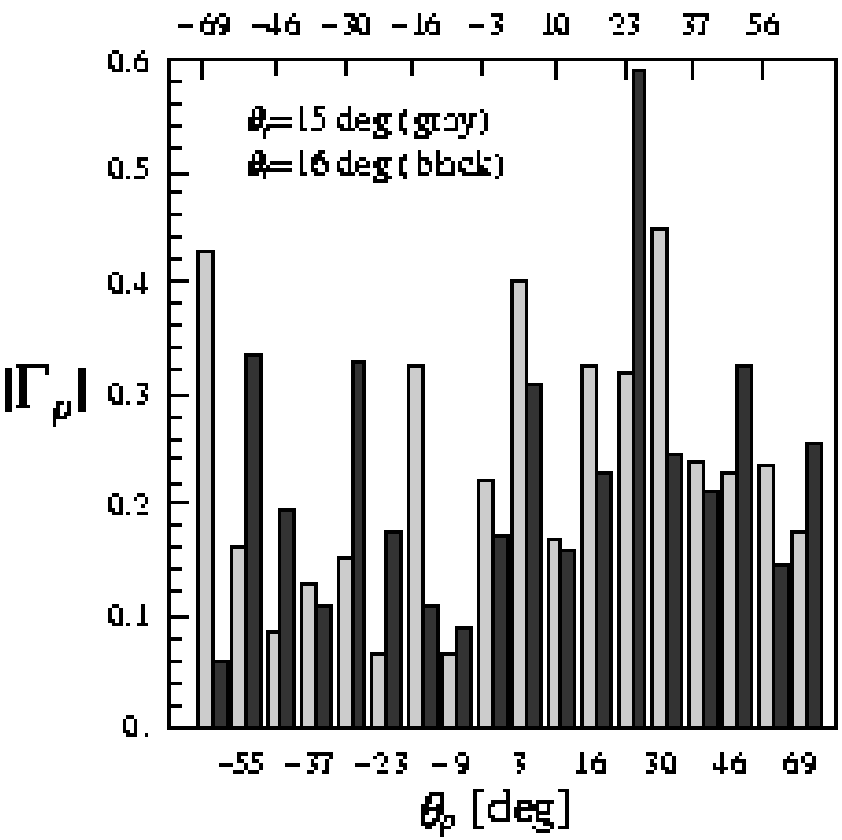


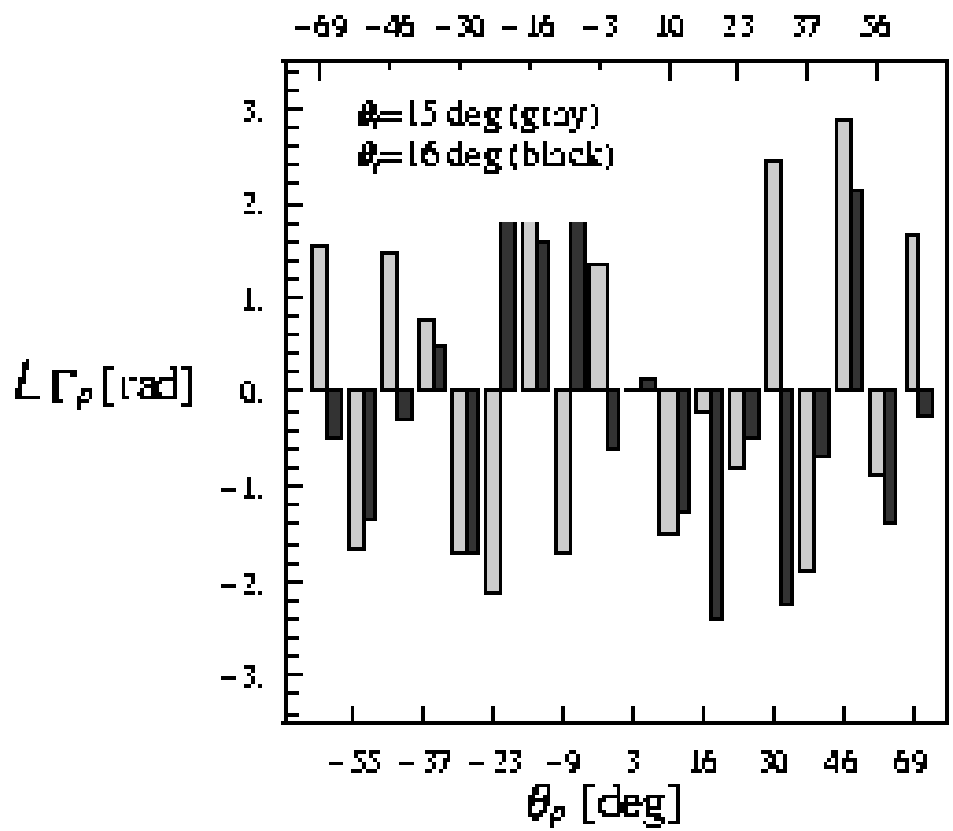
Full-wave simulations – Reflected field (cont'd)

Undulated surface with tapered dielectric topping

- Floquet coefficients depend smoothly on incidence angle if iff $a/\lambda << 1$;
- Floquet coefficients depend wildly on incidence angle iff $a/\lambda >> 1$;

$(a/\lambda=9,\ n(0)=15)$





Conclusions and open issues

- *New* paradigm in ray-chaotic boundary-value problem
  - Both internal and external,
  - Combined effect of both *geometrical* and *constitutive* effects.

  [Castaldi *et al.*, *Phys. Rev. Lett.*, submitted 2002]

- Ray-chaos hallmarks in the full-wave regime entail *nontrivial* EM response [Castaldi *et al.*,*J. Opt. Soc. Am.*, submitted 2002].

- Potential applications (currently under consideration)
  - Radar countermeasures,
  - Smart EM absorbers.

- Open issues
  - Characterize the deterministic/stochastic transition (work in progress, in collaboration with Prof. L.B. Felsen).