A UNIFORM ASYMPTOTIC DYADIC GREEN'S FUNCTION FOR A SEMI-INFINITE DIELECTRIC MEDIUM

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Abstract

This paper is concerned with the classical problem of the Green's function of an elementary source embedded in a dielectric half-space. An asymptotic evaluation is proposed of the pertinent Sommerfeld's integrals, which makes use of new canonical functions for the TM case. The final asymptotics gradually blends into the non uniform ray solution far from critical angle of GO incidence. Numerical results prove the accuracy of the formulation for both TE and, most remarkably, TM case.

INTRODUCTION

An accurate asymptotic evaluation of the dyadic Green's function for a semi-infinite dielectric medium is useful in the framework of an integral equation analysis of embedded objects. Non-uniform asymptotic (i.e., ray) description of the problem were presented by various authors [1], [2]. The most common way is to derive the asymptotics from the well known Sommerfeld spectral integral, by resorting to a standard SDP technique, which leads to a field representation in terms of space-wave and lateral-wave contributions, the latter associated to an incidence angle greater than the critical angle. The space-wave ray, which coincides at the first asymptotic order with the Geometrical Optics (GO) reflected field, emerges from a specular point, in accordance with the Snell's law; the lateral wave (LW) ray originates at point which is associated to the critical angle of incidence, propagates at grazing and next leaves the surface to reach the observation point. When the two points from which space and lateral wave originates merge together, both types of waves exhibit a transition and the simple non-uniform asymptotics fails. The asymptotic parameterization in the Sommerfeld's spectral plane is given in terms of branch-point and saddle-point coalescence. The absence in literature of a robust uniform asymptotics for treating this transition have motivates a further investigation. Indeed, while the asymptotic solution uniformly valid everywhere is given for TE case in the classical work of Bleistein [3], the TM case is not yet fully exploited. The difficulty that arises from this latter case resides in the presence of a Leaky-wave pole in an improper Riemann sheet. This pole is never captured by the SDP deformation for any observation point; however, its vicinity to the branch-point affects the ordinary saddle-point evaluation during the saddle-point/branch-point coalescence. This renders invalid, especially for high dielectric contrast, a conventional uniform asymptotics based on mapping saddle-point/branch-point interactions onto canonical parabolic cylinder functions. In this paper, while treating the TM case with the use of a new canonical function, also modifies the TE case with respect to the work of Bleinstein in order to obtain an asymptotic representation which blends more gradually and clearly into the non uniform GO-plus-lateral-wave ray description [1].

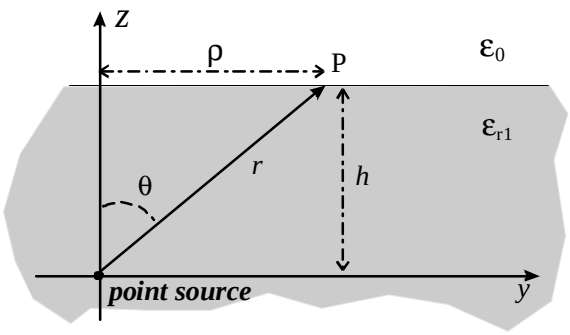

Fig. 1 *Geometry of semi-infinite dielectric slab illuminated by a point source*

FORMULATION

For the sake of simplicity, the formulation presented here is restricted to the case of an observation point at the dielectric interface However, the asymptotic strategy can be easily generalized to arbitrary observation point. The geometry of the problem is presented in Fig. 1. The point source is placed at distance h from the inteface, in the denser medium. A local (x,y,z) coordinate system is introduced with its origin at the dipole and its z-axis perpendicular to the lens interface is introduced; a spherical (r,θ,ϕ) and a cylindrical (ρ,ϕ,z) coordinate systems are also defined. The vertical electric and magnetic Hertz-Debye potentials can be written in terms of Sommerfeld integrals

$$I_{TE,TM} = \frac{1}{4\pi}\int_{-\infty}^{\infty} \tilde{I}_{TE,TM}\left(k_\rho\right) H_0^{(2)}(\rho k_\rho) e^{-jk_{z1}h} k_\rho dk_\rho \tag{1}$$

in which $k_{z0} = \sqrt{k_0^2 - k_\rho^2}$, $k_{z1} = \sqrt{k_1^2 - k_\rho^2}$, where $k_1 = \sqrt{\varepsilon_{r1}} k_0$ and $k_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ are the wavenumber of the dielectric medium. The subscript TE and TM denote TE and TM solution of the z-trasmission line problem in the spectral domain [Felsen]. This solution depends for the type of source considered; by referring for simplicity to an electric horizontal dipoles, we have

$$\tilde{I}_{TE} = \varsigma_0 \frac{k_0}{k_{z0} + k_{z1}} = \frac{\zeta_0}{k_0(1-\varepsilon_{r1})}(k_{z0} - k_{z1}); \quad \tilde{I}_{TM} = \frac{\varsigma_0}{k_0} \frac{k_{z1} k_{z0}}{\varepsilon_{r1} k_{z0} + k_{z1}}; \tag{2a,b}$$

where for the TE case the second equality has been obtained just multiplying and dividing by $(k_{z0} - k_{z1})$.

ASYMPTOTIC EVALUATION OF THE SOMERFELD INTEGRALS

By approximating the Hankel function in (1) with its expression for large value of its argument and using the angular change of variable $k_\rho = k_1 \sin\omega$, with $k_{z1} = k_1 \cos\omega$, the original real axis contour of the k_ρ plane is mapped into the contour $\Omega \equiv \left(-\frac{\pi}{2} - j\infty, \frac{\pi}{2} + j\infty\right)$. The integral is then transformed into

$$I_{TE,TM} = c' \int_\Omega \cos\omega \sqrt{\sin\omega}\ \tilde{I}_{TE,TM}\left(k_1 \sin\omega\right) e^{-jk_1 r \cos(\omega - \vartheta)} d\omega \quad ; \quad c' = \frac{k_1}{4\pi}\sqrt{\frac{2k_1}{\pi\rho}} e^{j\pi/4} \tag{3}$$

The ω plane, shown in Fig. 2, exhibits a branch point singularity on $\theta = \theta_c$ (critical angle), coming from $k_\rho = k_0$ in the radial spectral plane, while the branch point corresponding to $k_\rho = k_1$ disappears, due to the choice $k_{z1} = k_1 \cos\omega$. The saddle point $k_\rho = k_{\rho s}$ is transformed in the saddle point at $\omega = \theta$, where θ is the angle which individuates the observation point. In order to isolate asymptotic contributions, the original path is deformed into a Steepest Descent Path (SDP) through the saddle point θ. The asymptotic evaluation of this integral leads to the GO contribution. When $\theta \le \theta_c$, no branch-cut contributions are captured (Fig. 2a), while for $\theta > \theta_c$ (Fig. 2b) an additional integration around the branch cut must be included, which leads to the lateral wave contribution. For θ far away from θ_c $(k_{\rho s}$ far from $k_0)$ the GO contribution and the lateral wave contribution are distinct. In this case a non-uniform asymptotic expression can be easily obtained by expanding the amplitude function close to the critical points. This approximation fails for a branch point close to a saddle point, i.e. for observation aspects close to the critical angle, thus imposing a more sophisticated asymptotics for describing the transition field.

By using the above contour deformation, the original integral in (3) can be split as

$$I_\xi = I_{\Gamma_{SDP}} + U(\theta - \theta_c) I_{\Gamma_b}; \quad I_\Gamma = c' \int_\Gamma f_\xi(\omega) e^{-jk_1 r \cos(\omega - \vartheta)} d\omega; \tag{4}$$

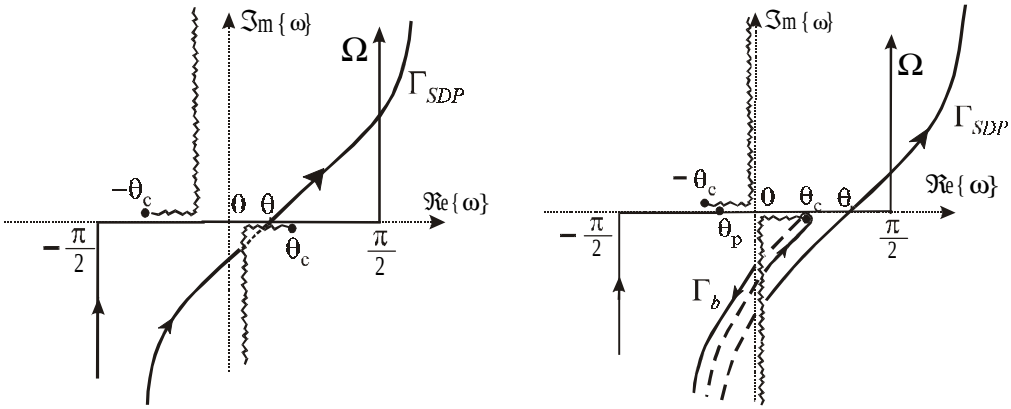

Fig. 2 *Integration paths in the spectral angular plane. (a) Angle of incidence smaller than the critical angle; (b) angle of incidence greater than the critical angle. The improper pole θ_p occurs only for TM case and is located on the bottom Riemann sheet associated to both branch-cuts.*

where $f_\xi(\omega) = \cos\omega\sqrt{\sin\omega}\ \tilde{I}_\xi^\eta(k_1 \sin\omega)$, $U(x)$ is a stepped unit function (equal to zero for $x \le 0$ and equal to one for $x > 0$), and $I_\Gamma = I_{\Gamma_{SDP}}\ or\ I_{\Gamma_b}$ where Γ_{SDP} denotes the SDP path and Γ_b is the path around the branch cut (see Fig. 2). By performing the substitution $\cos(\omega-\theta)=1-js^2$, which maps the saddle point in s=0 and the branch point θ_c in $s_c = \sqrt{-j2}\sin[(\theta_c-\theta)/2]$, the integrals in (4) becomes

$$I_\Gamma = c' e^{-jk_1 r}\int_\gamma F_\xi(s) e^{-k_1 r s^2} ds\ ;\ F_\eta(s) = f_\xi(\omega)\frac{d\omega}{ds} \quad (5)$$

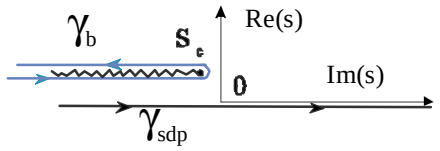

Fig. 3 *Complex s-plane and relevant contours*

and γ is the mapping of Γ in the s-plane (see Fig. 3). In particular, for $\Gamma=\Gamma_{SDP}$, $\gamma=\gamma_{SDP}$ is along the real axis of the s-plane, and for $\Gamma=\Gamma_b$, $\gamma=\gamma_b$ circumvents the branch cut at s_c. This branch is such that the phase of s-s_c varies from $-\pi$ and π in the top Riemann sheet. Depending on the case TE or TM, a different canonical mapping is chosen, as described next.

TE case. The integrand is mapped into the following form:

$$F_{TE}(s) \approx A + B\sqrt{s-s_c} + C(s-s_c)\sqrt{s-s_c} \quad (6)$$

where the constants A, B and C are found in such a way to match the integrand at the critical points of the s-domain. To clarify the above, let us consider the contributions to $F_{TE}(s)$ which comes from k_{z0} and k_{z1} in (2a) and let us denote them by $F_{TE0}(s)$ and $F_{TE1}(s)$, respectively. Since in the first mapping in the ω-plane the branch corresponding to k_{z1} =0 disappears, $F_{TE1}(s)$ does not present any branch point. Thus, the constant A is found by the value obtained by $F_{TE1}(s)$ at the saddle point; i.e. $A=F_{TE1}(0)$. The constant B and C are found indeed by matching the value of $F_{TE0}(s)$ at both the saddle point and the branch point; i.e., by solving the system $\lim_{(s\to s_c)}(F_{TE0}(s_c)/\sqrt{s-s_c}) = B$ and $F_{TE0}(s) = B\sqrt{-s_c} + C(-s_c)\sqrt{-s_c}$. Using (6) in (5) leads to a solution written in terms of the following canonical functions

$$a_n(s_0,\Omega,\gamma) = \int_\gamma (s-s_0)^n e^{-\Omega s^2} ds \quad ; \quad \mathrm{n} = 1/2,\ 3/2,\ \ \gamma=\gamma_{\mathrm{sdp}}, \gamma_{\mathrm{b}} \quad (7)$$

which can be rewritten in terms of ordinary one-parameter cylinder parabolic functions for both the contours $\gamma=\gamma_{\mathrm{sdp}}$ and $\gamma=\gamma_{\mathrm{b}}$.

TM case. The mapping in (6) does not work for the TM case, because of the presence of an additional pole on the bottom Riemann sheet (RS) of the s-plane which is located at $s_p = \sqrt{-j2}\sin[(\theta_p-\theta)/2]$ with $\theta_p = \sin^{-1}(1/\sqrt{\varepsilon_r+1})$. Although this pole cannot be never captured by the previous contour deformation, its presence may affect the asymptotic when the branch point is close to the saddle point, especially for high dielectric contrast. For this reason, a different mapping is suggested:

$$F_{TM}(s) \approx \frac{A\sqrt{s-s_c}}{\sqrt{s-s_c}+\sqrt{s_p-s_c}} + B\sqrt{s-s_c}(s-s_c) \quad (8)$$

where the constant A, and B are found in such a way to match the integrand by matching the value of $F_{TM}(s)$ at both the saddle point and the branch point; i.e., by solving the system:

$$A = \lim_{(s \to s_c)} \sqrt{s_p - s_c}\, F_{TM}(s) / \sqrt{s - s_c} \quad \text{and} \quad F_{TM}(s) = \frac{A\sqrt{-s_c}}{\sqrt{-s_c} + \sqrt{s_p - s_c}} + B(-s_c)\sqrt{-s_c}\ .$$

The term which multiplies B implies within (5) the same special function as that in (7) (with n=3/2), while for the term multiplying A , a new canonical function needs, which is defined as

$$b_n\left(s_0, s_p, \Omega, \gamma\right) = \int_\gamma \frac{\sqrt{s - s_0}}{\sqrt{s - s_0} + \sqrt{s_p - s_0}} e^{-\Omega s^2} ds \tag{9}$$

The numerical evaluation of this function, which is not more complicated than conventional cylinder parabolic functions, has been investigated and will be discussed during the session.

NUMERICAL EXAMPLES

Numerical examples are shown in Fig. 4**,** which are relevant to a medium with relative permittivity ε_{r1} = 4 (critical angle of incidence θ_c = 30 degrees). The numeric integration of (1) (reference curve, dashed line) is compared with the saddle-point GO-ray approximation (dotted line) and the uniform asymptotic evaluation (continuous line) for $h = 4\lambda_0$, where λ_0 is the free-space wavelength. Also shown are the curves of I_{SDP} (dashed-dotted line) and I_B (dashed-double-dotted line) separately. The good agreement shown in this case (which is remarkable for the TM case) has been also found for various situations of source position and dielectric contrast.

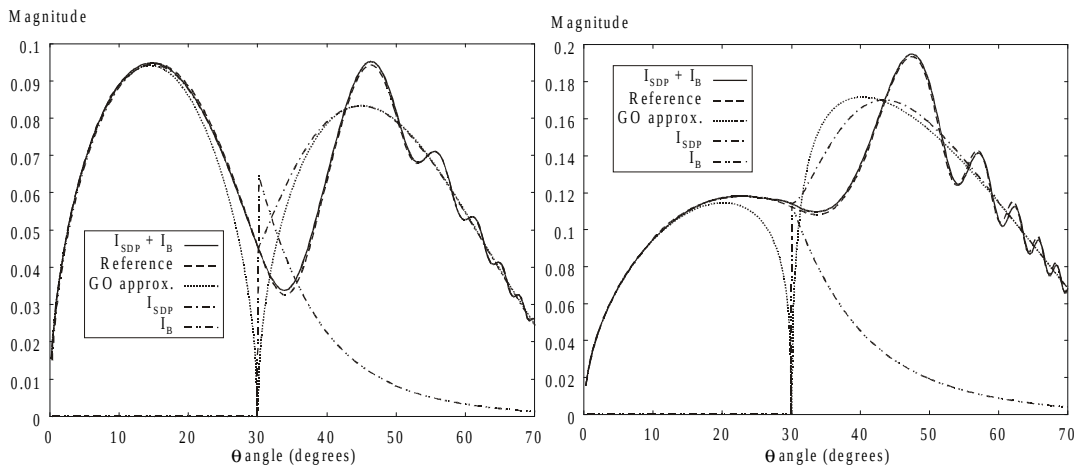

Fig. 4. *Green's function of the potential at the interface. (a) TE case (b) TM case*

CONCLUDING REMARKS

A uniform asymptotic evaluation or the potential Green's function for a semi-infinite dielectric medium as been presented. A new canonical function have been introduced for the TM case, which accounts for the presence of a pole on the improper Riemann sheet. In deriving the asymptotics, each spectral amplitude has been approximated by a rational function, which preserve the exact value of the spectrum at both saddle-point and branch-points. The final outcome is a representation which gradually blends into the non-uniform ray-field structure out of the transition region were the lateral wave merges into the space wave.

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