AN EFFICIENT FORMULATION FOR NUMERICAL EVALUATION OF THE GREEN'S FUNCTIONS OF LARGE PLANAR PHASED ARRAYS

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Abstract

The present work illustrates a formulation to evaluate spectral integral that represent the array Green's function (AGF) of large finite planar phased array for observation points close to the array contour. The procedure is based on the AGF representation in terms of a double spectral integral, whose numerical evaluation takes advantage of the exponential attenuation on a proper integration path. Thanks to this convergence properties, the final algorithm is numerically accurate, stable and more efficient with respect to the individual element summation.

INTRODUCTION

The array Green's function (AGF) represents the basic constituent for the full-wave description of electromagnetic radiation from rectangular periodic arrays and scattering from periodic surfaces. In modeling the performance of such structures, one of the main objectives is the reduction of the often prohibitive numerical effort that accompanies the full-wave analysis based on integral equation, which is structured around the ordinary individual element Green's function. For a periodic array, this AGF is composed of the sum over the individual dipole contributions. Under certain assumptions, the integral equation can be restructured around the active Green's function, which is the field collectively radiated by an array of elementary dipoles. A recent series of papers have investigated about the description of the AGF in terms of propagating and evanescent Floquet Waves (FWs) together with corresponding FW-modulated diffracted fields, which arise from FW scattering at the array edges and vertexes [1], [2]. The above mentioned formulation, being based on an asymptotic evaluation of the constituent radiation integrals, fails when observing too close to edge and vertexes. The present work is intended to illustrate a complementary strategy for treating observation point also close to the array contour, as implicated by a full-wave scheme. The same scheme, which imposes the solution of an integral equation, needs in most of cases the evaluation of the field at the array plane, i.e., where the boundary conditions are imposed. Consequently, the formulation will be carried out for this case, which is however the most useful and, at the same time, critical from numerical point of view. The procedure is based on the AGF representation in terms of double spectral integral, whose numerical evaluation takes advantage by the exponential attenuation on a proper integration path. Thanks to this convergence properties the final algorithm is numerically accurate and more efficient with respect to the individual element summation. Although example are presented here for the case of free-space Green's function, the same procedure can be modify to account for stratified media. A sample calculation is included to demonstrate the accuracy of this numerical method.

FORMULATION

A rectangular planar array linearly phased and uniformly illuminated in amplitude can be rigorously decomposed in terms of planar angular sector arrays [2]. Thus, it is not restrictive to start from a sectoral periodic array of linearly phased dipoles located in the z_1, z_2-plane (Fig.1 a). The inter-element spatial period along z_1 and z_2 is given by d_1 and d_2, and the inter-element phase gradient by g_1 and g_2, respectively. All dipoles are oriented along the unit vector $\hat{p}$ (a bold character denotes a vector quantity, and a caret ^ denotes a unit vector). A time dependence exp*(jwt)*, is understood and suppressed. In [1] it is shown that the electromagnetic vector field at the observation point $\boldsymbol{r} = z_1\hat{z}_1 + z_2\hat{z}_2 + y\hat{y}$ can be represented in the spectral (k_{z_1}, k_{z_2}) form of the free space Green's function

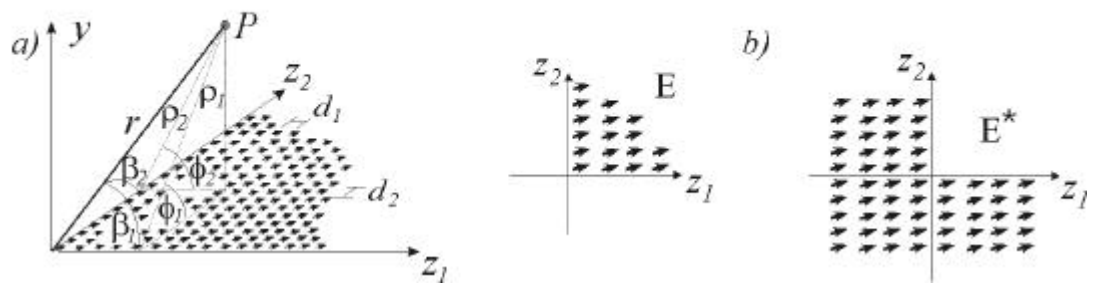

Figura 1 *a) Geometry of the planar sectoral array of parallel dipoles oriented along a direction* $\hat{p}$ *. b) Geometry of the actual and the complementary array; this latter is used for extending the procedure for* $z_1>0$, $z_2>0$ *.*

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{8\boldsymbol{p}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\overline{G}}(k_{z_1}, k_{z_2}) \cdot \hat{p} \, \frac{B_1(k_{z_1}) B_2(k_{z_2}) e^{-j(k_{z1} z_1 + k_{z2} z_2 + k_y y)}}{k_y} dk_{z_1} dk_{z_2} \tag{1}$$

where $k_y = \sqrt{k^2 - k_{z_1}^2 - k_{z_2}^2}$ with $\Im m\{k_y\} < 0$ for $k^2 - k_{z_2}^2 < k_{z_1}^2$ and $k_y>0$ for $k^2 - k_{z_2}^2 > k_{z_1}^2$ on the top Riemann sheet of the complex k_{z_i} -plane (see [2]). The spectral dyadic Green's function $\overline{\overline{G}}(k_{z_1}, k_{z_2})$ is that associated to the single dipole source; for the sake of simplicity, we will refer to a free-space environments, but the formulation is valid also for stratified media, providing to include the proper Green's function in the spectral treatment. For free-space $\overline{\overline{G}}(k_{z_1}, k_{z_2}) = (-z/k)(k^2 \overline{\overline{I}} - \boldsymbol{kk})$ with $\overline{\overline{I}}$ denoting the unit dyadic and $\boldsymbol{k} = k_{z_1} \hat{z}_1 + k_{z_2} \hat{z}_2 \pm k_y \hat{y}$. The integrand in (1) also includes the closed form spectral functions [2]

$$B_i(k_{z_i}) \equiv \frac{1}{1 - e^{jd_i(k_{z_i} - g_i)}} \qquad i = 1,2. \tag{2}$$

which exhibit *q*- and *p*-indexed poles in the (k_{z_1}, k_{z_2}) plane, located at $k_{z_1} = k_{z_1,q} = g_1 + 2p\, q / d_1$, $k_{z_2} = k_{z_2,p} = g_2 + 2pp / d_2$, $q, p = 0, \pm 1, \pm 2 \ldots$. As seen in [2], these functions arise from the collective radiation of all dipoles on a sectoral array.

As mentioned before, it is particular important to investigate about the field at *y=0*, because of potential use of the AGF in integral equations. In this case, (1) can be restructured into a sequence of complex integral as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{8\boldsymbol{p}^2} \int_{-\infty}^{\infty} B_2(k_{z_2}) \boldsymbol{D}(k_{z_2}) \, e^{-jk_{z2} z_2} dk_{z_2} \tag{3}$$

$$\boldsymbol{D}(k_{z_2}) = \int_{-\infty}^{\infty} \frac{B_1(k_{z_1}) \overline{\overline{G}}(k_{z_1}, k_{z_2}) \cdot \hat{p}}{\sqrt{k_b^2 - k_{z_1}^2}} \; e^{-jk_{z_1} z_1} dk_{z_1} \tag{4}$$

with $k_b = \pm\sqrt{k^2 - k_{z_2}^2}$. In (3), the integrand function contains poles at $k_{z_2} = k_{z_2,p}$ due to $B_2(k_{z_2})$. In (4) the integrand function has the branch-point singularities at $k_{z_1} = \pm k_b$ and poles at $k_{z_1} = k_{z_1,q}$ due to $B_1(k_{z_1})$ (see (2)). Both k_{z_1}, k_{z_2} integration circumvent the poles in clockwise sense [2]. We note that the residues of these poles in both variables represent the FW associated to the infinite array. The location of branch points and branch cuts, with respect to the real-axis integration path in the k_{z_1} -plane, is found by introducing small fictitious losses ($\Im m\{k\} = 0^-$)

which are eventually removed [1]. For the sake of convenience, we start from the case $z_1<0$ and next we extend the procedure to $z_1>0$.

A. Region: $z_1<0$. First, the integration (4) in the spectral variable k_{z_1} ("internal" integration), is numerically evaluated. Then, the outcome is used in the subsequent integration (3), ("external integration") in the spectral variable k_{z_2}. The "internal" integration is carried out by deforming the original integration path along the branch cut $\Re e\{k_{z_1}\} = -\Re e\{k_b\}$ (Fig.2 a b). Note that the vanishing of the integration contribution along C_1, C_2 imposes $z_1<0$, which is the reason of the initial assumption. Furthermore, due to the clockwise indentation along the poles, no residue has to be included during the contour deformation. This latter fact may be understood by observing that when the observation point is located outside the array region, the description of the field not include FWs. Using the contour parity, "internal" integration, deformed along the branch cut, can be reduced to an integral from 0 to ∞ with the change of variable $k_{z_1} = -k_b + jkx$, thus leading to

$$\boldsymbol{D}(k_{z_2}) = 2k e^{jz_1\sqrt{k^2-k_{z_2}^2}} \int_0^\infty \frac{\overline{\overline{G}}(k_{z_1}, k_{z_2}) \cdot \hat{p}}{(1-e^{jd_1(-k_b+jkx-g_1)})\sqrt{k^2x^2+2jk_b kx}} \; e^{kxz_1} dx \tag{5}$$

The numerical integration can be easily performed for $z_1<0$; thanks to the exponential attenuation, the k_{z_1} integration can be practically truncated at $x \cong 2l/|z_1|$ with negligible errors. The external integration (3) along the k_{z_2} real axes is carried out by subdividing the axes into intervals with amplitude Dk (Fig. 2 c), which contains or less $k_{z_2,p}$ poles; in this way the electric field can be separated as $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}'(\boldsymbol{r}) + \boldsymbol{E}''(\boldsymbol{r})$, respectively. For intervals without poles, the integrand function is continuous and the integration can be performed by ordinary routine for numerical integration. In intervals which contains poles the integrand function is regularized by adding and subtracting from $B_2(k_{z_2})$ the function $j/d_2(k_{z_2}-k_{z_2,p})$, where j/d_2 is the residues at the poles $k_{z_2,p}$. The integrand regularized function is then expressed by means of Taylor expansion around $k_{z_2,p}$-poles, i.e., $1/(1-e^{jd_2(k_{z_2}-g_2)}) - j/d_2(k_{z_2}-k_{z_2,p}) \cong 1/2 + d_2(k_{z_2}-k_{z_2,p})/j12$. In this way $\boldsymbol{E}'$ can be separated in three terms considering the Taylor expansion of $\boldsymbol{D}(k_{z_2}) \cong \boldsymbol{a} + \boldsymbol{b}(k_{z_2}-k_{z_2,p})$ around $k_{z_2,p}$-poles, where $\boldsymbol{a} = \boldsymbol{D}(k_{z_2,p})$, $\boldsymbol{b} = [\boldsymbol{D}(k_{z_2,p}+\Delta k/2) - \boldsymbol{D}(k_{z_2,p}-\Delta k/2)]/\Delta k$. Note that the terms coming from the integration of the three above contributions can be evaluated in a closed form.

B. Region: $z_1>0$, $z_2<0$. In analogy with the first case, this range can be covered by interchanging the order of integration. This means in practice to perform the interchange of subscripts *1* with *2* in (3)-(5). Note that the range $z_1<0$, $z_2<0$ may be treated by using the one or the other order of sequential integration, the most convenient choice being dictated by the ratio z_1/z_2.

C. Region: $z_1>0$, $z_2>0$. For this case we define a complementary dipole array (Fig.1 b) which summed to the original sectoral array forms an infinite periodic array. For this infinite array, the field (denoted by $\boldsymbol{E}^{FW}$) can be obtained in terms of Floquet Wave expansion as [1]

$$\boldsymbol{E}^{FW} = \sum_{p,q=-\infty}^{\infty} \boldsymbol{E}_{pq}^{FW} = \sum_{p,q=-\infty}^{\infty} \overline{\overline{G}}(k_{z_1,q}, k_{z_2,p}, k_{ypq}) \frac{e^{-j(k_{z_1,q}z_1 + k_{ypq}y + k_{z_2,p})}}{2d_1 d_2 k_{ypq}} \tag{6}$$

with $k_{ypq} = \sqrt{k^2 - k_{z_1,q}^2 - k_{z_2,p}^2}$. From the linearity of the solution, it is evident that $\boldsymbol{E}^* + \boldsymbol{E} = \boldsymbol{E}^{FW}$ where $\boldsymbol{E}^*$ is the field of the complementary array. Thus $\boldsymbol{E} = \boldsymbol{E}^{FW} - \boldsymbol{E}^*$ which holds for all value of (z_1,z_2). Using this solution, for $z_1>0$, $z_2>0$, we reconstruct the remaining portion of space not covered by the formulation, provided to know $\boldsymbol{E}^*$. To obtained $\boldsymbol{E}^*$ we can considered the complementary array (see Fig.1 b) as composed by three sectoral array localized for $(z_1<0, z_2>0)$, $(z_1<0, z_2<0)$ and $(z_1>0, z_2<0)$, respectively.

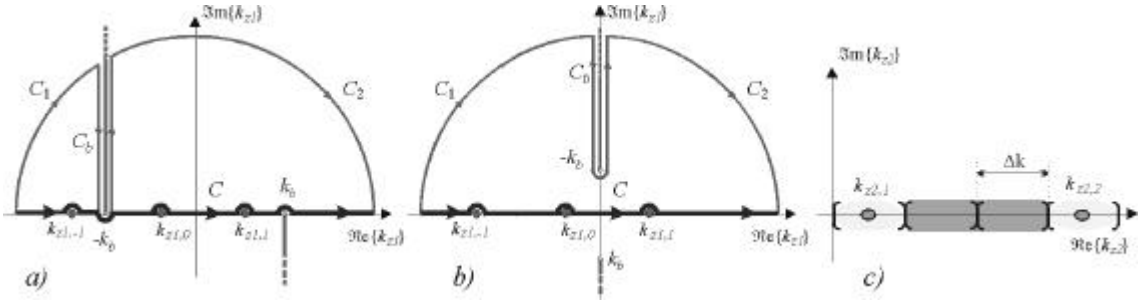

Figure 2. *Topology of the* k_{z_1} *plane; the poles* $k_{z_1,q}$ *are located below the original integration path; branch points occur at* $k_{z_1} = \pm k_b = \pm\sqrt{k^2 - k_{z_2}^2}$ *; (a)* $k^2 > k_{z_2}^2$ *, branch cut along the lines* $\Re e\{k_{z_1}\} = \Re e\{\pm k_b\}$, $\Im m\{k_{z_1}\} > (<) 0$ *;(b)* $k^2 < k_{z_2}^2$ *, branch cut along the lines* $\Re e\{k_{z_1}\} = 0$, $\Im m\{k_{z_1}\} > (<) \Im m\{\pm k_b\}$ *c)Possible subdivision of the* k_{z_2} *external spectral variable into intervals which contain or less FW poles.*

With the same procedure illustrated previously, the field $\boldsymbol{E}^*$ can be obtained as a sum of the three contribution radiated by the three sectoral array that form the complementary array.

NUMERICAL RESULTS AND CONCLUSION

Fig. 3. shows numerical results for the electric field on the array plane for a rectilinear scan range shown in the inset. The array is composed by 50x50 dipoles that are oriented along z_1 with periodicity equal to 0.8λ and phase $g_1 = g_2 = 0$. The result are compared with a element by element summation (dotted line) of individual dipole contribution. Note that for calculating the above plots, the external integration has been performed from (*-3k, 3k*), which means three times the visible region. The internal (k_{z_1})-integration (5) is performed from zero to $2l / |z_1|$. The procedure presented here for equi-amplitude excitation, can be generalized for tapered illumination as well as for stratified media.

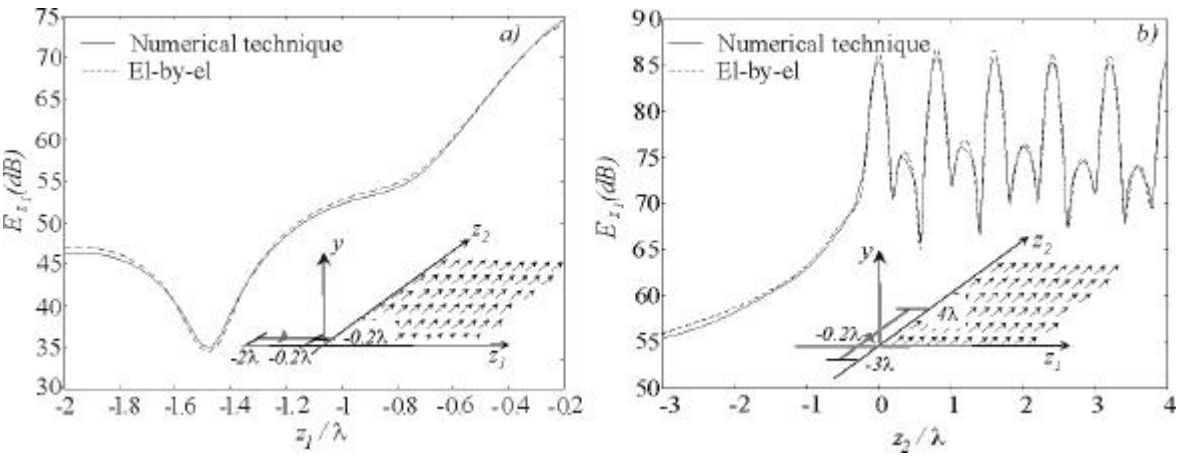

Figure 3. *Comparison between element by element solution (dotted line) with numerical technique (dashed line).* Z_1 *component of the electric field for the rectilinear scan shown in the inset.*

References

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