Physical MoM Preconditioning for modeling Large Periodic Arrays

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Abstract

A preconditioner for the full-wave analysis (e.g. MoM) of large periodic one-dimensional arrays of elements with general shape (not necessarily planar) is developed, based on the physical properties induced by periodicity for the infinite and truncated array problems. Using the preconditioner dramatically reduces the number of iterations needed to solve the linear system (e.g. the discretized electric field integral equation). Moreover, the right hand side of the preconditioned equation can be used directly as an approximate solution for initial design purposes.

I. Introduction

Computational resources needed for solving large array problems by the method of moments (MoM) include large times for matrix filling and solving as well as large memory requirements to store the MoM matrices. The *fill time* can usually be reduced by using asymptotic evaluations of the mutual coupling between array elements. Furthermore, due to geometrical translational symmetry, it is possible to reuse coupling results for translated element pairs. To reduce the *solution time*, various algorithms have been developed that use periodicity-induced physical properties. For example, the method of [1], [2] uses an a priori estimate of the fields scattered by truncated arrays, which behave as Floquet-modulated-diffracted fields [3], to construct global basis functions.

Here, we analyze array antennas or periodic structures with arbitrary excitation, and focus on reducing the solution time using accurate and physically-based MoM preconditioners in an iterative solution to find the current on the array elements. The preconditioner is constructed using properties induced by the periodicity. The general idea is that in many practical cases, most of the elements comprising the large array behave much like those of an infinite array, and this information is used to construct the preconditioner. As with any preconditioner, the solution procedure is accurate because the original integral equation is maintained. Our formulation is based on an identity relating the impedance and admittance operators for the infinite and finite periodic structures. The preconditioned problem is particularly advantageous when various solutions are desired for the same geometrical configuration. Once the preconditioner is constructed, the system is usually solved in only a few iterations, permitting the evaluation of various parameters such as element currents, element input impedances, active input impedances, reduction in array performance due to element failures, coupling between element pairs in the presence of the finite array, etc. The preconditioner is independent of the excitation, and therefore the preconditioned problem can also be efficiently used to analyze scanning properties of an array without reevaluating the MoM matrix or preconditioner.

In the present study we assume a one/two-dimensional periodic array configuration that is infinite along one direction and finite in the other, with general element geometry (not necessarily planar). This implementation characterizes most of the truncation effects in actual array problems, and edge effects are analyzed with a reduced computational effort. It is shown that with preconditioning, matrix condition numbers often fall by several orders of magnitude. Furthermore, when the MoM system is solved using a stabilized biconjugate gradient algorithm, the number of iterations may be reduced from several hundred to a handful. As stated, in many practical cases the first iteration (the preconditioned right hand side of the equation) is sufficiently accurate that it may be used as an approximate solution of the MoM system of equations.

II. Statement of the Problem

Consider an array as shown in Fig. 1, infinitely long in the y direction (either uniform or periodic) and truncated in the x direction. The excitation can be arbitrary, i.e., the illumination could be a plane wave or other incident field, or voltages may be applied directly to some of the array elements. For the 2D array in Fig. 1b, the only assumption is that in the y direction the excitation is equi-amplitude with a possible

phase progression, reducing the problem to a two-dimensional analysis of a one-dimensional array. The numerical analysis thus reduces to consideration of a single row of elements along x, since the Green's function accounts for the infinite dimension of the array along y. The standard electric field integral equation (EFIE) for the finite array in x is written as

$$Z^{aa} I^{a} = V^{a} , \tag{1}$$

which determines the current I^a on the actual array elements (the central row in the x direction, at $y = 0$). The moment matrix Z^{aa}, whose elements represent interactions between basis and testing function pairs on the central row (using the Green's function of a y-periodic linear array of point sources), is assumed to be partitioned into blocks representing interactions between pairs of array elements on the central row. We focus here on the construction of an effective preconditioner, i.e., a multiplicative matrix that approximates the inverse of the MoM impedance matrix.

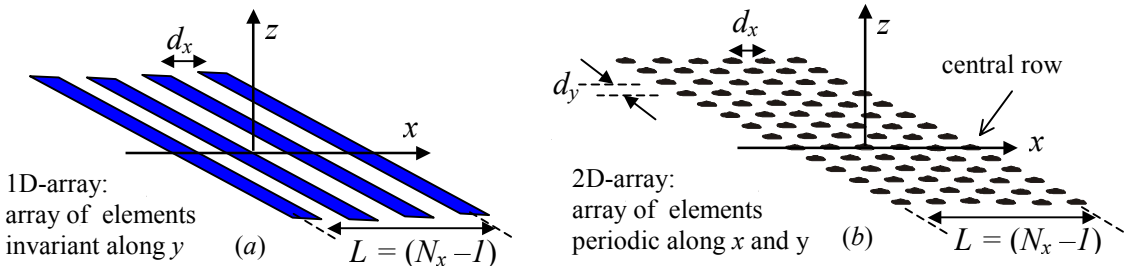


Figure 1. Arrays that are infinite in the y direction and finite along x. (a) An array of elements that are uniform in the y direction. (b) An array of elements that are infinite and periodic in y, with a uniform progressive phase shift in the y-direction.

III. Formulation

Consider first the two dimensional infinite periodic array formed by extending the actual array in the x direction in both directions. The elements of the extended array constitute the *virtual array*. Superscripts a and v denote the *actual* and *virtual* array, respectively (see the cross-sectional geometry in Fig. 2). For this new problem, after proper partitioning, the EFIE and its inverse can be symbolically written as

$$\begin{bmatrix} Z^{aa} & Z^{av} \\ Z^{va} & Z^{vv} \end{bmatrix} \begin{bmatrix} I^{a} \\ I^{v} \end{bmatrix} = \begin{bmatrix} V^{a} \\ V^{v} \end{bmatrix} \rightarrow \begin{bmatrix} I^{a} \\ I^{v} \end{bmatrix} = \begin{bmatrix} Z^{aa} & Z^{av} \\ Z^{va} & Z^{vv} \end{bmatrix}^{-1} \begin{bmatrix} V^{a} \\ V^{v} \end{bmatrix} \equiv \begin{bmatrix} Y^{aa} & Y^{av} \\ Y^{va} & Y^{vv} \end{bmatrix} \begin{bmatrix} V^{a} \\ V^{v} \end{bmatrix} . \tag{2}$$

These matrix equations relate the actual and virtual array problems. Note that the EFIE for the actual array (1) can be obtained from (2) by requiring that the current I^v on the virtual array vanish. Combining the impedance and admittance matrices for the infinite array we derive the identities

$$Y^{aa} Z^{aa} = U - Y^{av} Z^{va} , \qquad \left[Z^{aa} \right]^{-1} = \left[U - Y^{av} Z^{va} \right]^{-1} Y^{aa} , \tag{3a,b}$$

which have the following interpretations. First, when Y^{aa} is used as a preconditioner for Z^{aa} in (1), it regularizes the hypersingular part of Z^{aa} since $Y^{aa}Z^{aa}$ is equal to the identity operator U plus a diffracted operator $Y^{av}Z^{va}$ that contains no array element self interactions. Indeed, as shown in the figure below, when $Y^{av}Z^{va}$ operates on a current I of the actual array, it produces voltage excitations $V^v=Z^{va}I$ on the virtual array that in turn produce changes in current $Y^{av}V^v$ back on the actual array. Furthermore, the current $Y^{av}Z^{va}I$ decays away from the array truncation as a diffracted current (ignoring, for simplicity, the possible presence of modes supported by the structure). Second, (3b) relates the admittance (matrix inverse) operator $[Z^{aa}]^{-1}$ for the *finite* array to the sub-block Y^{aa} of the admittance matrix for an *infinite* array, with truncation effects represented by physical interactions between the actual and virtual array.

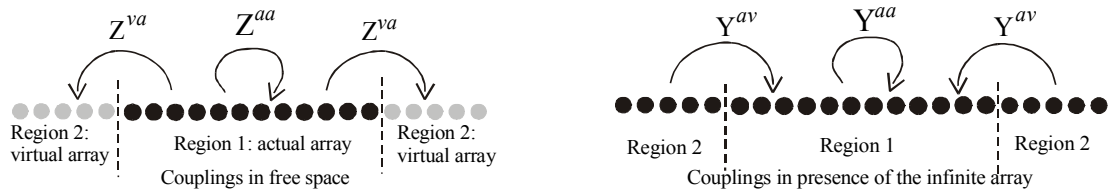


Figure 2. Impedance and admittance interactions between actual array and virtual array. The impedances are evaluated in free space. The admittances are evaluated in the presence of the infinite array, thus involving the solution of the entire infinite array problem with specific excitations.

Note that Y^{aa} is the restriction of the *infinite* array admittance matrix to the finite array. Away from the edges of the array, the term $Y^{av}Z^{va}I$ is small since it represents a diffraction contribution that decays away from the edges. To avoid computing a matrix inverse, we therefore approximate $[U-Y^{av}Z^{va}]^{-1} \approx U+Y^{av}Z^{va}$, which, again, approaches U for currents far from the array edges. The inverse of the MoM matrix thus may be approximated as

$$\left[Z^{aa}\right]^{-1} \approx Y^{P}, \qquad Y^{P} \equiv \left[U + Y^{av}Z^{va}\right]Y^{aa} \quad . \tag{4a,b}$$

Further approximations may be made to any of the matrices on the RHS to simplify the preconditioner. Practically, for example, it is likely that the domains and/or ranges of these matrix operators must be windowed. In the following we suppose the array consists of N_x array elements in the x direction. The current vector I^a and the known excitation vector V^a are partitioned as $I^a=[[I_1], \ldots[I_{Nx}]]^T$, $V^a=[[V_1], \ldots[V_{Nx}]]^T$, and the Z^{aa} matrix is partitioned accordingly. The preconditioner need not include all the interactions since the region of significant influence of an array element usually extends to just a few nearby elements. Thus only a few interactions need be taken into account in constructing the terms Y^{aa} and $Y^{av}Z^{va}$. In the next section we provide a possible algorithm to construct these terms; a number of additional variations of the approach may also be considered. A simpler version of the preconditioner can be obtained by neglecting the diffractive effects due to $Y^{av}Z^{va}$, i.e., $[Z^{aa}]^{-1} \approx Y^{aa}$.

IV. Construction of the Preconditioner from the Infinite Array Solution

If the array is formed by N_x array elements in the x direction, the matrix Y^{aa} is formed of N_x sub-blocks, though in practice we use a smaller number $2E+1$ of sub-blocks, accounting only for the interactions between the nearest E array elements to the left and right of an element. The matrix Y^{av} has infinite dimension, however, we approximate the virtual array using only F elements nearest the array edges. In practice, we may choose $F = E$ so that all the sub-blocks of Y^{av} replicate those of Y^{aa}. Often, E and F can be as small as two or three. Analogous arguments hold for the evaluation of Z^{va}, which reuses previously evaluated blocks of Z^{aa}. A key feature of the approach is the fact that matrices Y^{aa} and Y^{av} can be evaluated from the solution of the infinite array problem as follows. We denote by $[Z^{\infty}(k_x)]$ the $N_e \times N_e$ element MoM matrix for a unit cell of the *infinite array problem* (infinite in x) with progressive interelement phase shift k_x; $[Y^{\infty}(k_x)]$ is its inverse. Furthermore, let $[Y_n^{\infty}]$ denote the element-to-element mutual coupling admittance matrix between two array elements of the *infinite* array, n cells apart (n can be positive or negative). Matrices $[Y^{\infty}(k_x)]$ and $[Y_n^{\infty}]$ are related by the Fourier series transform pair

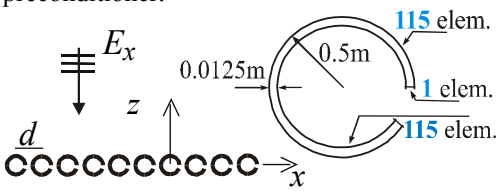
$$\left[Y^{\infty}\left(k_x\right)\right] = \sum_{n=-\infty}^{\infty}\left[Y_n^{\infty}\right]e^{-jk_x n d_x} = \left[Z^{\infty}\left(k_x\right)\right]^{-1}, \left[Y_n^{\infty}\right] = \frac{d_x}{2\pi}\int_{-\pi/d_x}^{\pi/d_x}\left[Y^{\infty}\left(k_x\right)\right]e^{jk_x n d_x}dk_x \ , \tag{5a,b}$$

which may be used to obtain $[Y_n^{\infty}]$ once $[Y^{\infty}(k_x)]$ is computed for the infinite array. Note that $[Y^{\infty}(k_x)]$ is periodic in k_x with period $2\pi/d_x$, and that (unlike impedance matrices) $[Y_n^{\infty}] \neq [Y_n]$, i.e. the element mutual admittance matrices for infinite and finite arrays are not identical. We summarize the computation of Y^{aa} and Y^{av} as follows: (a) Determine the active impedance matrix $[Z^{\infty}(k_x)]$ for a unit cell of the infinite array problem for $-\pi/d_x \le k_x \le \pi/d_x$, using the periodic Green's function. (Though not discussed here, the computation of the Green's function is accelerated using the Ewald method. See [4] for the 1D array (Fig. 1a) and [5] for the 2D array (Fig. 1b).) This is a small array of size $N_e \times N_e$. (b) Compute $[Y^{\infty}(k_x)]$ by inverting $[Z^{\infty}(k_x)]$. (c) Compute $[Y_n^{\infty}]$ for all required n using (5b). (d) Construct Y^{aa} as the block-Toeplitz

matrix having $[Y_n^{\infty}]$ as the nth block from the diagonal. Y^{av} is constructed using the same matrix sub-blocks $[Y_n^{\infty}]$.

V. Numerical Examples (1D Array of Open Cavities Invariant along z)

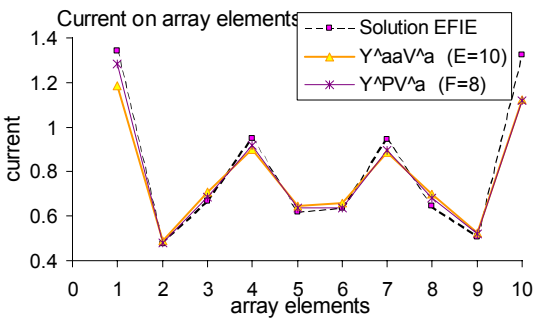
An array of 10 thick open cavities is illuminated by a perpendicular TE_z plane wave at f = 150MHz, which is *far* from cavity internal resonance frequencies. As depicted in the figure below, each array element is finely discretized, with a total of 115 elements on the inner and outer surfaces and 1 element on each end face, for a total of 232 subelements on each array element. Note that the small condition number (CN) of the preconditioned EFIE, and the dramatic reduction in the number of iterations when using Y^{aa} as a preconditioner.



Equation Solved (f=150MHz)		CN	Iterations
$Z^{aa}I^a = V^a$		7695	649
$Y^{aa}Z^{aa}I^a = Y^{aa}V^a$	E=10	5	2
$Y^{p}Z^{aa}I^a = Y^{p}V^a$	E=10 (F=8)	5	2

Further improvement is obtained using the preconditioned equation $(U\text{-}Y^{av}Z^{va})I^a = Y^{aa}V^a$ (see identity (3a)) since the replacement of $Y^{aa}Z^{aa}$ by $U\text{-}Y^{av}Z^{va}$ eliminates the computation of the self interactions; effectively, we analytically extract the identity matrix U from $Y^{aa}Z^{aa}$. In a second test for this array of cavities, plane wave excitation was replaced by *single element excitation* using a voltage generator placed on top of the 4th element (from the left). Again, the system was solved in only two iterations using the previously computed matrices Z^{aa} and Y^{aa}. It is important to note that for a numerical integration of (5b), which samples the Brillouin zone, using only 21 sample points yields a preconditioner that still only requires three iterations for the solution of the MoM equation. This shows that the preconditioner is not "expensive" to compute even for large array problems (see [6] for a physical explanation).

Approximation of Inverse by Preconditioner. The two preconditioners Y^{aa} and Y^{P} are now used to approximate the inverse matrix $[Z^{aa}]^{-1}$. Currents directly evaluated as $I^a \approx Y^{aa}V^a$, or as $I^a \approx Y^{p}V^a$, without solving a linear system, are compared with the "exact" solution provided by the EFIE $Z^{aa}I^a = V^a$ solved numerically. The current sampled on the top-most point of each element is plotted below. Use of the simple preconditioner $I^a \approx Y^{aa}V^a$, which contains low-order diffraction effects, provides good results, while the more elaborate preconditioner $I^a \approx Y^{p}V^a$ slightly improves the current estimate. It is expected that arrays that support modes (surface waves) would require the more elaborate Y^{p} to accurately estimate mode reflection at the edges.



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