

# MODELLI CIRCUITALI DI ANTENNE A GEOMETRIA FRATTALE

## CIRCUIT MODELS OF FRACTAL-GEOMETRY ANTENNAS<sup>1</sup>

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### Abstract

*This work develops some recursive circuit models for a simple but efficient description of the electromagnetic features of fractal-shaped antennas whose geometrical layout is based on the Sierpinski triangle topology. In particular, two classes of models are studied, the 'esoskopik' ones, that characterise the input impedance and reflection coefficients through a multiport equivalent network only at external access points, and the 'endoskopik' ones, that can furnish an equivalent circuit-level description of the whole geometrical layout.*

### INTRODUCTION

It has been widely shown that the autosimilarity properties of the fractal geometries can be successfully applied to the project of multiband antennas [1]. In particular, some devices based on the Sierpinski triangle topology have a log-periodic behaviour as far as both the input parameters and the radiation diagrams are concerned [2].

In this work we aim at developing recursive circuit models for an efficient description of the electromagnetic (EM) features of the fractal structure (in particular the frequency dependence of the input reflection coefficient and of the input impedance). The recursive topology is analogous to the intrinsic nature of the generation process of the fractal structure, that is based on the action of a particular *Iterated Function System* (IFS), i.e., a feedback system whose output relevant to a given operating step becomes the input relevant to the following step [3].

### FORMULATION

#### A- Esoskopik models

In the 'esoskopik' modelization, the 'Pre-Fractal' (PF) structure, at every finite step  $n \in N$  of the iterative generation process, is characterised by means of an analogy with a Multiport Network (MN). For the specific case of the triangular Sierpinski topology (see Fig. 1), in Fig. 2 is depicted the  $n$ -th order iteration of the generation process, consisting in the construction of the  $(n+1)$ -order PF, represented by the impedance MN matrix  $Z^{(n+1)}$ , by using three replicas of the  $n$ -order PF, all with impedance<sup>2</sup> MN matrix  $Z^{(n)}$ , in two different versions that use as constitutive building-blocks 2-port unbalanced networks and 3-port balanced networks, respectively.

The model is completely defined by the parameters  $Z^{(0)}$ , that is the *starting point* matrix, and  $f$ , that is the matrix function that represents the iteration procedure, i.e.:

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<sup>2</sup> For a  $m$ -port network the constitutive relation is  $ZI=V$ , where  $Z$  is the  $m \times m$  impedance matrix,  $I$  and  $V$  are the  $m \times 1$  port current and voltage matrices, respectively. In particular, the  $j$ -th diagonal element of  $Z$  represents the input impedance at the  $j$ -th port when the other ports are at a zero current state.

$$Z^{(n+1)} = f(Z^{(n)}) \quad (1)$$

For instance, in the 2-port case the iterative function  $f$  has the following form:

$$f\left(\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\right) = \begin{bmatrix} 2z_{11} + \zeta + \frac{(z_{11} + \zeta)^2}{z_{12} + z_{21} - 2(z_{11} + z_{22}) - \zeta} & z_{12} - \frac{(z_{11} + \zeta)(z_{22} + \zeta)}{z_{12} + z_{21} - 2(z_{11} + z_{22}) - \zeta} \\ z_{21} - \frac{(z_{11} + \zeta)(z_{22} + \zeta)}{z_{12} + z_{21} - 2(z_{11} + z_{22}) - \zeta} & 2z_{22} + \zeta + \frac{(z_{22} + \zeta)^2}{z_{12} + z_{21} - 2(z_{11} + z_{22}) - \zeta} \end{bmatrix} \quad (2)$$

If we introduce the matrix function  $f_{(n)}$  that directly furnishes the  $n$ -th order PF  $Z^{(n)}$  from the starting point  $Z^{(0)}$ , i.e.  $f_{(n)}(Z^{(0)}) = Z^{(n)}$ , the *fractal* object is ‘reached’ by taking the limit  $f_{(\infty)}(Z^{(0)}) = \lim_{n \rightarrow \infty} f_{(n)}(Z^{(0)})$ .

The classical Sierpinski triangular topology (Fig. 1b) is obtained by taking as starting point  $Z^{(0)}$  of the iterative procedure given by Eq. (2) the MN of a Symmetric Triangle (ST) that we assume composed by bipoles of impedance  $\zeta_0$  (see Fig. 3):

$$Z^{(0)} = Z_{st} = \zeta_0 \cdot \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (3)$$

and by letting the interconnection bipole<sup>3</sup>  $\zeta=0$ . In such case, we can show that:

$$Z_{st}^{(n)} = f_{(n)}(Z_{st}) = (5/3)^n \cdot Z_{st}, \quad (4)$$

i.e. the structure of a ST network is identically replicated except for the scaling factor 5/3; therefore, its inverse 3/5 can be assumed as the renormalization factor in the iterative process even for a generic starting point matrix; therefore, we shall use, as constitutive IFS, the function  $\tilde{f}(Z^{(n)}) = 3/5 \cdot f(Z^{(n)})$  instead of the non-normalized IFS given by Eq. (2), that would furnish diverging impedance values for  $n \rightarrow \infty$ .

By applying the Caccioppoli-Banach theorem, it can be shown that  $\tilde{f}$  has contractive features with fixed-point  $Z_{ts}$ ; i.e. a high-order PF object, even if composed by non-symmetrical elementary blocks ( $Z^{(0)} \neq Z_{ts}$ ), presents, from an external view-point, a behaviour quantitatively analogous to a ST structure:  $Z^{(n)} \rightarrow \alpha \cdot Z_{ts}$  for  $n \rightarrow \infty$  (where  $\alpha$  is a constant depending on  $Z^{(0)}$ ): see Fig. 4.

## B- Endoscopic models

The ‘endoscopic’ modelization is directly obtained by associating a linear, permanent, bipole to every link of the graph  $\Gamma_n$  representing the topology of the  $n$ -th order Sierpinski gasket PF<sup>4</sup>. The iterative composition procedure is the following one:

*Initial Step:* the 0-th order PF Network (PFN) is a simple triangle whose vertices  $\{v_0^0, v_0^1, v_0^2\}$  are enumerated in base-3 as  $\{‘0_3’, ‘1_3’, ‘2_3’\}$ . The 3×3 adjacency matrix is:

$$A^{(0)} \equiv A^{[0]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad (5)$$

<sup>3</sup> Alternative significant models can be obtained by assuming  $\zeta = \zeta_0$  (Sierpinkij ‘gasket’ topology, see Fig. 1a), or, more in general,  $\zeta = \xi \cdot \zeta_0$ , with  $0 \leq \xi \leq 1$ .

<sup>4</sup> For the ‘gasket’ topology (Fig. 1a),  $\Gamma_n$  has  $V_n$  vertices and  $L_n$  links, with:  $V_n = 3^{n+1}$ ,  $L_n = (3^{n+1} - 1) \cdot 3/2$ ; instead for the classical Sierpinski triangle (Fig. 1b) we have:  $V_n = 3 \cdot (3^n - (3^n - 1)/2)$ ,  $L_n = 3^{n+1}$ .

Analogously, if we assume identical values  $\zeta_0$  for the link impedances, the  $3 \times 3$  admittance matrix<sup>5</sup>, with respect to an external ground vertex  $v_G$ , is:

$$Y^{(0)} \equiv Y^{[0]} = (\zeta_0)^{-1} \cdot \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad (6)$$

*Iterative Step:* the  $n$ -th order PFN is obtained by replacing each  $k$ -vertex  $v_{n-1}^k$  of the  $(n-1)$ -th order PFN ( $k=0,1,\dots,3^n-1$ ) with a triangle  $\{v_n^{k,0}, v_n^{k,1}, v_n^{k,2}\}$ ; if  $v_{n-1}^k$  is enumerated by means of a base-3 string with  $n$  digits, i.e.:  $v_{n-1}^k \equiv [\kappa^0, \kappa^1, \dots, \kappa^{n-1}]$ , its ‘sons’ of the  $n$ -th order PFN shall be mutually distinguished by an additional base-3 digit, i.e.:

$$v_n^{k,l} \equiv [\kappa^0, \kappa^1, \dots, \kappa^{n-1}, l] \quad , l = 0, 1, 2 \quad (7)$$

From an alternative point of view, the  $n$ -th order graph  $\Gamma_n$  can be obtained by replicating and mutually connecting three  $(n-1)$ -th order graphs  $\Gamma_{n-1}^0, \Gamma_{n-1}^1, \Gamma_{n-1}^2$  (see Fig. 1a); therefore the  $3^{n+1} \times 3^{n+1}$  adjacency matrix  $A^{(n)}$  is composed by a triple diagonal block composition of the lower order  $3^n \times 3^n$  matrix  $A^{(n-1)}$ , that takes into account the internal connections of the single  $(n-1)$ -th order sub-regions, and a suitable interconnection matrix  $V^{[n]}$ , that takes into account the three links that connect different subregions (e.g. links  $\lambda_n^0, \lambda_n^1, \lambda_n^2$  in Fig. 1a for the  $0 \rightarrow 1$  step):

$$A^{(n)} = \bigoplus_3 A^{(n-1)} + V^{[n]}; \quad (8)$$

$$V^{[n]} = \begin{cases} V_{ij}^{[n]} = \delta_{i,(3^n-1)/2} \delta_{j,3^n} + \delta_{i,3^n-1} \delta_{j,2 \cdot 3^n} + \delta_{i,2 \cdot 3^n-1} \delta_{j,(5 \cdot 3^n-1)/2}, & 0 \leq i < j \leq 3^{n+1} - 1 \\ V_{ij}^{[n]} = V_{ji}^{[n]}, & 0 \leq j < i \leq 3^{n+1} - 1 \\ V_{ii}^{[n]} = 0, & 0 \leq i \leq 3^{n+1} - 1 \end{cases} \quad (9)$$

( $\delta$  is the Kronecker symbol).

The iterated composition is analogous for the admittance matrix:

$$Y^{(n)} = \bigoplus_3 Y^{(n-1)} + \Upsilon^{[n]} \quad (10)$$

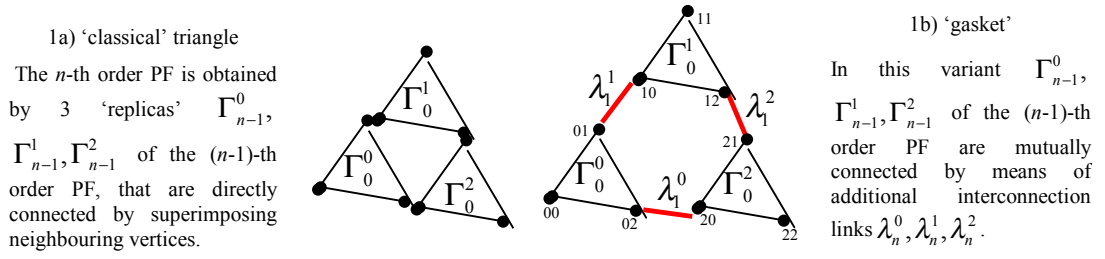
$$\Upsilon^{[n]} = \begin{cases} \Upsilon_{ij}^{[n]} = -\zeta^{-1} \cdot \left( \delta_{i,(3^n-1)/2} \delta_{j,3^n} + \delta_{i,3^n-1} \delta_{j,2 \cdot 3^n} + \delta_{i,2 \cdot 3^n-1} \delta_{j,(5 \cdot 3^n-1)/2} \right), & 0 \leq i < j \leq 3^{n+1} - 1 \\ \Upsilon_{ij}^{[n]} = \Upsilon_{ji}^{[n]}, & 0 \leq j < i \leq 3^{n+1} - 1 \\ \Upsilon_{ii}^{[n]} = \zeta^{-1} \cdot \left( \delta_{i,(3^n-1)/2} + \delta_{i,3^n} + \delta_{i,3^n-1} + \delta_{i,2 \cdot 3^n} + \delta_{i,2 \cdot 3^n-1} + \delta_{i,(5 \cdot 3^n-1)/2} \right), & 0 \leq i \leq 3^{n+1} - 1 \end{cases} \quad (11)$$

where  $\zeta$  is the impedance associated to the interconnection links (see note 3).

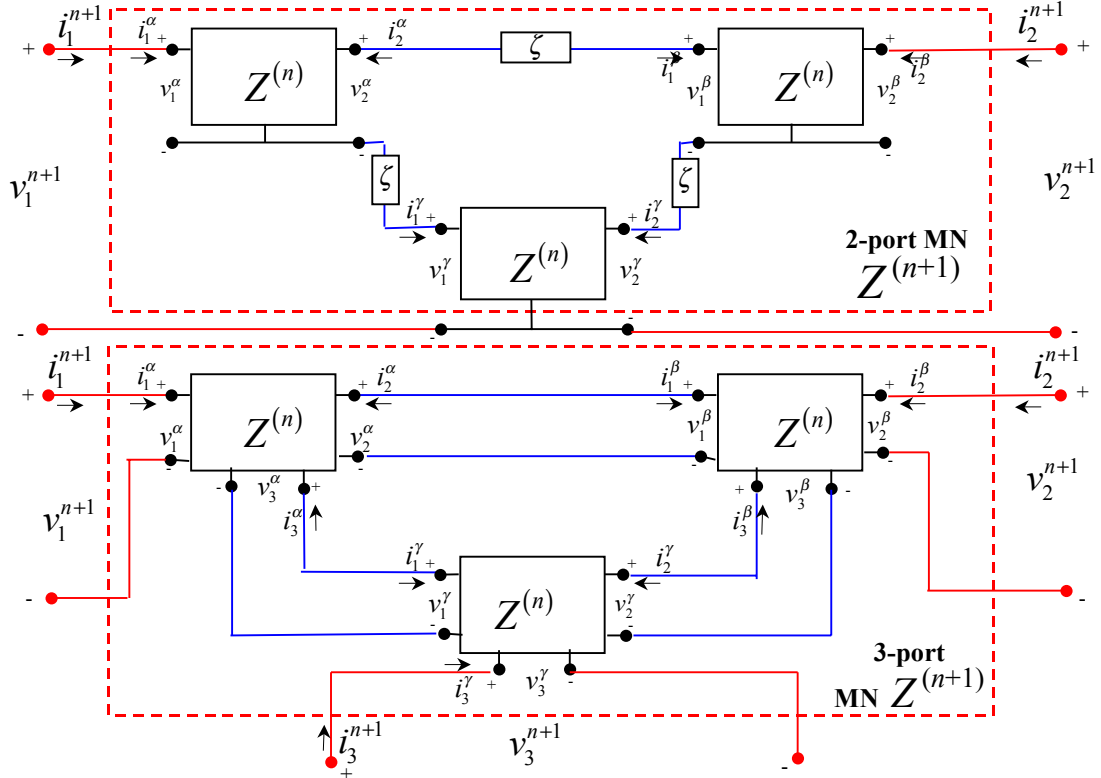
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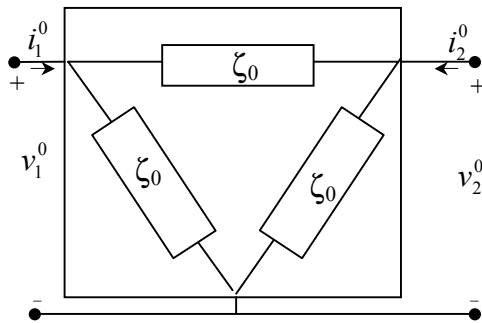
<sup>5</sup> As is well known:  $Y_{kk}$ =sum of the admittances relevant to the links converging to the  $k$ -th vertex;  $Y_{kh}$ =admittance relevant to the link between the  $h$ -th and the  $k$ -th vertices. The  $k$ -th diagonal element of the inverse matrix  $Y^{-1}$  represent the input impedance of the port realized between the  $k$ -th vertex and the external ground vertex  $v_G$  (with all other ports left open, see note 2), that can model the physical situation of the fractal antenna fed by means of an external power source (e.g. coaxial cable) connected at the  $k$ -th vertex of the planar layout.



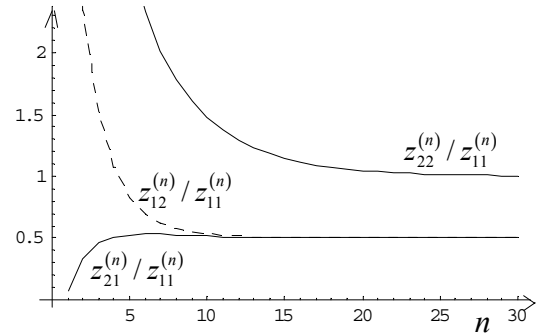
**Fig. 1:** The Sierpinski topologies under analysis. Example of the  $0 \rightarrow 1$  step of the iterative construction procedure for the classical triangular shape (1a) and the *gasket*-shaped variant (1b).



**Fig. 2 -** The  $n$ -th order iteration of the esoscopic modelizations with 2-port unbalanced networks (*up*) and with 3-port balanced networks (*down*).



**Fig. 3 -** The symmetric triangular inner composition of the 'starting' 0-th order 2-port network  $Z^{(0)} = Z_{st}$  for the Sierpinski triangular topologies.



**Fig. 4 -**  $n$ -th order MN ratios  $z_{22}^{(n)}/z_{11}^{(n)}$ ,  $z_{12}^{(n)}/z_{11}^{(n)}$  and  $z_{21}^{(n)}/z_{11}^{(n)}$  vs.  $n$  for  $Z^{(0)} = \zeta_0 \begin{bmatrix} 2 & 14 \\ -1 & 21 \end{bmatrix}$ ,  $\zeta_0=0$ ; for high  $n$  the ratios tend to the typical values of the ST network, i.e. 1, 0.5 and 0.5, respectively.