DIFFRACTION BY A MOVING WEDGE: COMPARISON OF ANALYTIC SOLUTION AND NUMERICAL TECHNIQUES

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Abstract

In this work the problem of electromagnetic wave scattering by a perfectly conducting wedge in uniform translating motion is treated on the ground of the Frame Hopping Method, by means of a plane-wave spectra representation approach: the solution is found both through an exact analytical procedure and through the application of two different numerical techniques that can be used for solving diffraction problems by moving objects with arbitrary shape.

I-INTRODUCTION

The solution of electromagnetic (EM) diffraction problems by scattering objects in uniform translating motion can be obtained on the ground of the covariance properties of the electrodynamic 4-tensors. In fact the expressions of the EM field, as measured by observers in different inertial reference frames in relative motion, can be interrelated by means of the relativistic Lorentz Transformation formulae. On the ground of such transformations it is possible to establish the *Frame Hopping Method* [1], that was pioneered by A. Einstein in the foundation work of the Special Relativity Theory [2] for the 1D problem of EM reflection by a translating mirror. The major drawback of the FHM for application-oriented problems is that the relativistic covariance transformations become rather cumbersome for a generic EM field when expressed in terms of the usual 3-vector notation. On the other hand, in the case of an EM Plane-Wave (PW) and, by virtue of linearity, of a PW-expandable EM field it is possible to express the relativistic transformation formulae in terms of simple alteration rules for wave parameters, see [3,4]. Therefore, two simplification techniques can be developed: the *PW Local Approximation* (PWLA) and the *PW Integral Expansion* (PWIE). In the first case [1], once the scattering problem is solved in the co-moving frame through the customary motionless approach, the EM field is approximated by means of a single plane-wave term for every far-field location considered; then the Lorentz transformation to the laboratory frame are performed by operating on such single plane-wave terms through the parameter alteration rules. In the second case [3,4], the solution in the co-moving frame has to be found in term of a PW integral representation, so that, by virtue of the linear nature of the relativistic covariance relation, the Lorentz transformation to the laboratory frame can be applied by operating on every plane-wave spectral component through the parameter alteration rules. The PWLA technique is formally simpler but its solutions are approximated and sufficiently accurate only in points which are far away from the trajectory of the target; on the other hand PWIE furnishes exact solutions.

In this work we apply the PWIE to the specific case of a Perfectly Electric Conducting (PEC) wedge which uniformly translates in vacuo, that has been previously treated by other authors only by means of the PWLA [5]. In particular, the PWIE of the scattered field is based on the Fourier integral representation of the Non-Integer Cylindrical Waves (NICW) [6,7]; in fact, the simple linear procedure of Lorentz-transforming one by one the terms involved in the expansion is not applicable to the Sommerfeld formula which is commonly used in the literature for representing wedge diffraction [8], because it consists in an integration over a complex spectral variable, whose non-real values furnish field terms to the integral summation which (unlike plane-waves) do not individually verify the

Helmholtz separability condition in vacuo, i.e. they are not particular solution of the Maxwell equations and then they are not Lorentz-covariant.
The analytical results have been used as a benchmark for testing two numerical techniques, the *Relativistic Physical Optics* (ROP) and the *Relativistic Moment Method* (RMM), that have been developed, on the ground of the FHM-PWIE approach, for studying diffraction problems by moving objects with an arbitrary geometry [9].

II-FORMULATION

We distinguish between two inertial reference frames, the laboratory frame Σ and the co-moving frame Σ', where the scattering PEC wedge W' appears at rest; the relative velocity is $bc\hat{\mathbf{z}}$, where $c = 1\big/\sqrt{m_0 e_0}$ is the light speed in vacuo, see [3,7,9]. For an observer in frame Σ the incident EM field, that is assumed to be an arbitrarily polarized mono-chromatic PW with angular frequency *w*, has the following expression:

$$\left[\boldsymbol{E}_\mathrm{i}, \boldsymbol{H}_\mathrm{i}\right]_{\mathbf{r},t} \overset{\mathrm{Re}}{=} \left[\mathbf{E}_\mathrm{i}, \mathbf{H}_\mathrm{i}\right]_{\mathbf{r}} \exp\left[-iwt\right] = \left[\bar{\mathbf{E}}_\mathrm{i}, \bar{\mathbf{H}}_\mathrm{i}\right] \exp\left[iw\left(c^{-1}\hat{\mathbf{k}}_\mathrm{i} \bullet \mathbf{r} - t\right)\right] \qquad (1)$$

where $\boldsymbol{E}_\mathrm{i}$, $\boldsymbol{H}_\mathrm{i}$ are the incident electric and magnetic fields, at position $\mathbf{r}=[x, y, z]$ and time *t*, respectively; $i = +\sqrt{-1}$; superscript 'Re' means that the real part of the entire right member has to be taken; $\hat{\mathbf{k}}_\mathrm{i} = \left[-\sqrt{1-\left(\hat{k}_\mathrm{i}^y\right)^2 - \left(\hat{k}_\mathrm{i}^z\right)^2},\ \hat{k}_\mathrm{i}^y,\ \hat{k}_\mathrm{i}^z\right]$ is the wave unit vector; $\bar{\mathbf{E}}_\mathrm{i}$ and $\bar{\mathbf{H}}_\mathrm{i} = i\sqrt{e_0/m_0}\left(\hat{\mathbf{k}}_\mathrm{i} \times \bar{\mathbf{E}}_\mathrm{i}\right)$ are constant complex vectors determining the polarization.

As a first step of the FHM, by applying Lorentz transformation to Eq. (1), we obtain the expression of the incident EM field as observed in the co-moving frame Σ' at position $\mathbf{r}'=[x', y', z']=[x, y, g\,(z-bct)]$ and time $t'=g\,(t-bc^{-1}z)$, ($g = +\sqrt{1-b^2}$) :

$$\left[\boldsymbol{E}_\mathrm{i}', \boldsymbol{H}_\mathrm{i}'\right]_{\mathbf{r}',t'} = \left[\mathbf{E}_\mathrm{i}', \mathbf{H}_\mathrm{i}'\right]_{\mathbf{r}'} \exp\left[-iw't'\right] = \left[\bar{\mathbf{E}}_\mathrm{i}', \bar{\mathbf{H}}_\mathrm{i}'\right] \exp\left[iw'\left(c^{-1}\hat{\mathbf{k}}_\mathrm{i}' \bullet \mathbf{r}' - t'\right)\right] \qquad (2)$$

with PW parameters altered according to the following rules [3,4].

$$w' = g\left(1 - b\,\hat{k}_\mathrm{i}^z\right)w \qquad (3), \quad \hat{\mathbf{k}}_\mathrm{i}' = \left(1 - b\,\hat{k}_\mathrm{i}^z\right)^{-1}\left[-g^{-1}\sqrt{1-\left(\hat{k}_\mathrm{i}^y\right)^2 - \left(\hat{k}_\mathrm{i}^z\right)^2},\ g^{-1}\hat{k}_\mathrm{i}^y, \hat{k}_\mathrm{i}^z - b\right] \qquad (4)$$

$$\left[\bar{\mathbf{E}}_\mathrm{i}', \bar{\mathbf{H}}_\mathrm{i}'\right] = \left(\hat{\mathbf{x}}\hat{\mathbf{x}} + g\left(\hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}\right)\right) \bullet \left[\bar{\mathbf{E}}_\mathrm{i}, \bar{\mathbf{H}}_\mathrm{i}\right] + gbc \times \left[m_0\bar{\mathbf{H}}_\mathrm{i}, -e_0\bar{\mathbf{E}}_\mathrm{i}\right] \qquad (5)$$

As a second step of the FHM, we solve the motionless diffraction problem in frame Σ' in order to obtain a PW expansion of the scattered EM field $\left[\boldsymbol{E}_\mathrm{s}', \boldsymbol{H}_\mathrm{s}'\right]$:

$$\left[\boldsymbol{E}_\mathrm{s}', \boldsymbol{H}_\mathrm{s}'\right]_{\mathbf{r}',t'} \overset{\mathrm{Re}}{=} \iint_{-\infty}^{+\infty} \left[\bar{\mathbf{E}}_\mathrm{s}', \bar{\mathbf{H}}_\mathrm{s}'\right]_{\hat{k}_\mathrm{s}'^y, \hat{k}_\mathrm{s}'^z} \exp\left[iw'\left(c^{-1}\hat{\mathbf{k}}_\mathrm{s}' \bullet \mathbf{r}' - t'\right)\right] d\hat{k}_\mathrm{s}'^y\, d\hat{k}_\mathrm{s}'^z \qquad (6)$$

where $\hat{\mathbf{k}}_\mathrm{s}' = \left[+\sqrt{1-\left(\hat{k}_\mathrm{s}'^y\right)^2 - \left(\hat{k}_\mathrm{s}'^z\right)^2},\ \hat{k}_\mathrm{s}'^y, \hat{k}_\mathrm{s}'^z\right]$ (for $x' > x'_{\min} = \min_{\mathbf{r}' \in W'}\left[\mathbf{r}' \bullet \hat{\mathbf{x}}\right]$) is the wave unit vector, and $\bar{\mathbf{E}}_\mathrm{s}', \bar{\mathbf{H}}_\mathrm{s}'$ are the PW Spectral Amplitude Coefficients (PWSACs) , see sec. II.1.

Finally, as a last step of the FHM, we apply the $\Sigma' \rightarrow \Sigma$ Lorentz transformations to Eq. (6) and obtain the PW expansion of the scattered field in the laboratory frame (for $x > x'_{\min}$):

$$\left[\boldsymbol{E}_\mathrm{s}, \boldsymbol{H}_\mathrm{s}\right]_{\mathbf{r},t} \overset{\mathrm{Re}}{=} \iint_{-\infty}^{+\infty} \left[\bar{\mathbf{E}}_\mathrm{s}, \bar{\mathbf{H}}_\mathrm{s}\right]_{\hat{k}_\mathrm{s}'^y, \hat{k}_\mathrm{s}'^z} \exp\left[iv\left(\hat{k}_\mathrm{s}'^z\right) \cdot \left(c^{-1}\hat{\mathbf{k}}_\mathrm{s} \bullet \mathbf{r} - t\right)\right] d\hat{k}_\mathrm{s}'^y\, d\hat{k}_\mathrm{s}'^z \qquad (7)$$

with the PW parameters obtained by the following rules [3,4].

$$v\left(\hat{k}_\mathrm{s}'^z\right) = g\left(1 + b\hat{k}_\mathrm{s}'^z\right)w' = g^2\left(1 + b\,\hat{k}_\mathrm{s}'^z\right)\left(1 - b\,\hat{k}_\mathrm{i}^z\right)w \qquad (8)$$

$$\hat{\mathbf{k}}_{\mathrm{s}} = \left(1+b\,\hat{k}_{\mathrm{s}}'^{z}\right)^{-1}\left[+g^{-1}\sqrt{1-\left(\hat{k}_{\mathrm{s}}'^{y}\right)^{2}-\left(\hat{k}_{\mathrm{s}}'^{z}\right)^{2}},\ g^{-1}\hat{k}_{\mathrm{s}}'^{y},\hat{k}_{\mathrm{s}}'^{z}+b\right] \tag{9}$$

$$\left[\bar{\mathbf{E}}_{\mathrm{s}},\bar{\mathbf{H}}_{\mathrm{s}}\right] = \left(\hat{\mathbf{x}}\hat{\mathbf{x}}+g\left(\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)\right)\bullet\left[\bar{\mathbf{E}}_{\mathrm{s}}',\bar{\mathbf{H}}_{\mathrm{s}}'\right]-gbc\times\left[m_0\bar{\mathbf{H}}_{\mathrm{s}}',-e_0\bar{\mathbf{E}}_{\mathrm{s}}'\right] \tag{10}$$

II-1 PWSACs EVALUATION

Analytical Solution – Firstly, the scattered EM field is expanded as a discrete summation of NICWs according to the *Radial Transmission Representation* [10];

$$\left[\boldsymbol{E}_{\mathrm{s}}',\boldsymbol{H}_{\mathrm{s}}'\right]_{\mathbf{r}',t'} \overset{\mathrm{Re}}{=} \sum_{k=-\infty}^{+\infty}\left[\boldsymbol{e}_{n_k}',\boldsymbol{h}_{n_k}'\right]NICW_{n_k}\left(w'c^{-1}\mathbf{r}'\right)\exp\left[-iw't'\right] \tag{11}$$

where n_k are non-integer indexes and $\left[\boldsymbol{e}_{n_k}',\boldsymbol{h}_{n_k}'\right]$ are vector constant coefficients depending on the wedge geometry and on the incident field features [7,10]; secondly, each n_k-order NICW is expressed in terms of a Fourier integral expansion (see [6,7] for details):

$$NICW_{n_k}\left(w'c^{-1}\mathbf{r}'\right)=\int\!\!\int_{-\infty}^{+\infty}\Xi_k\left(\hat{k}_{\mathrm{s}}'^{y},\hat{k}_{\mathrm{s}}'^{z}\right)\exp\left[iw'c^{-1}\mathbf{r}'\bullet\hat{\mathbf{k}}_{\mathrm{s}}'\right]d\hat{k}_{\mathrm{s}}'^{y}\,d\hat{k}_{\mathrm{s}}'^{z}\quad, \tag{12}$$

so that the exact expression of the PWSACs in Eq. (6) is obtained:

$$\left[\bar{\mathbf{E}}_{\mathrm{s}}',\bar{\mathbf{H}}_{\mathrm{s}}'\right]_{\hat{k}_{\mathrm{s}}'^{y},\hat{k}_{\mathrm{s}}'^{z}} = \sum_{k=-\infty}^{+\infty}\left[\boldsymbol{e}_{n_k}',\boldsymbol{h}_{n_k}'\right]\Xi_k\left(\hat{k}_{\mathrm{s}}'^{y},\hat{k}_{\mathrm{s}}'^{z}\right) \tag{13}$$

Numerical Techniques – On the ground of the Equivalence Theorem (ET), the PWSACs in Eq. (6) can be related to the expression of the total EM field $\left[\boldsymbol{E}',\boldsymbol{H}'\right]_{\mathbf{r}',t'}=\left[\boldsymbol{E}_{\mathrm{i}}'+\boldsymbol{E}_{\mathrm{s}}',\boldsymbol{H}_{\mathrm{i}}'+\boldsymbol{H}_{\mathrm{s}}'\right]_{\mathbf{r}',t'}=\mathrm{Re}\left\{\left[\mathbf{E}',\mathbf{H}'\right]_{\mathbf{r}'}\exp\left[-iw't'\right]\right\}$ on the $\partial W'$, see [11]:

$$\left[\bar{\mathbf{E}}_{\mathrm{s}}',\bar{\mathbf{H}}_{\mathrm{s}}'\right]_{\hat{k}_{\mathrm{s}}'^{x},\hat{k}_{\mathrm{s}}'^{y}} = \left(8p^{2}\sqrt{1-\left(\hat{k}_{\mathrm{s}}'^{y}\right)^{2}-\left(\hat{k}_{\mathrm{s}}'^{z}\right)^{2}}\right)^{-1}\left\{-\hat{\mathbf{k}}_{\mathrm{s}}'\times\oint_{\partial W'}\exp\left[-iw'c^{-1}\hat{\mathbf{k}}_{\mathrm{s}}'\bullet\mathbf{r}'\right]\hat{\mathbf{n}}'\times\left[\mathbf{E}',\mathbf{H}'\right]_{\mathbf{r}'}dS'+\right.$$

$$\left.+c^{-1}\left[\mathbf{I}-\hat{\mathbf{k}}_{\mathrm{s}}'\hat{\mathbf{k}}_{\mathrm{s}}'\right]\bullet\oint_{\partial W'}\exp\left[-iw'c^{-1}\hat{\mathbf{k}}_{\mathrm{s}}'\bullet\mathbf{r}'\right]\hat{\mathbf{n}}'\times\left[-\left(e_0\right)^{-1}\mathbf{H}',\left(m_0\right)^{-1}\mathbf{E}'\right]_{\mathbf{r}'}dS'\right\} \tag{14}$$

a) RMM: The total EM field on ∂W can be exactly evaluated through the following ET- based surface integral equation [11]:

$$\hat{\mathbf{n}}_0'\times\left[\tfrac{1}{2}\mathbf{E}'-\mathbf{E}_{\mathrm{i}}',\tfrac{1}{2}\mathbf{H}'-\mathbf{H}_{\mathrm{i}}'\right]_{\mathbf{r}_0'} = \hat{\mathbf{n}}_0'\times\left\{\nabla_0'\times\oint_{\partial W'}G\left(\mathbf{r}_0'-\mathbf{r}'\right)\hat{\mathbf{n}}'\times\left[\mathbf{E}',\mathbf{H}'\right]_{\mathbf{r}'}dS'+\right.$$

$$\left.+\left(iw'\right)^{-1}\left[\left(w'c^{-1}\right)^{2}\mathbf{I}+\nabla_0'\nabla_0'\right]\bullet\oint_{\partial W'}G\left(\mathbf{r}_0'-\mathbf{r}'\right)\hat{\mathbf{n}}'\times\left[-\left(e_0\right)^{-1}\mathbf{H}',\left(m_0\right)^{-1}\mathbf{E}'\right]_{\mathbf{r}'}dS'\right\} \tag{15}$$

where $G\left(\mathbf{r}'\right)=\left(4p\left|\mathbf{r}'\right|\right)^{-1}\exp\left[iw'c^{-1}\left|\mathbf{r}'\right|\right]$; $\hat{\mathbf{n}}_0'$, $\hat{\mathbf{n}}'$ are the outward unit normal vectors at points $\mathbf{r}_0',\mathbf{r}'\in\partial W'$, respectively; ∇_0' is the 'nabla' operator with respect to the Σ' space-coordinates at point $\mathbf{r}_0'$. Eq. (15) has to be solved together with the PEC condition $\hat{\mathbf{n}}'\times\mathbf{E}'\left(\mathbf{r}'\right)=\mathbf{0}$, $\mathbf{r}'\in\partial W'$). By numerically solving on the ground of the Moment Method [12], we can get an approximation of the total EM field on $\partial W'$, and then, through Eqs. (6) and (14), a PW expansion of the scattered EM field valid in the vacuum space, that are asymptotically exact as far as the discretization lattice is taken to the continuum limit [9].

b) ROP: On the other hand, the ROP assumes an *aprioristic* approximation of the total EM field on $\partial W'$ in order to compute the PWACs comparing in Eq. (6); in the case of point, for a PEC wedge, we let, see [9,13]:

$$\hat{\mathbf{n}}'\times\left[\mathbf{E}',\mathbf{H}'\right]_{\mathbf{r}'} = \begin{cases}\left[\mathbf{0}, 2\times\mathbf{H}'_{\mathrm{i}}\right]_{\mathbf{r}'} & :\mathbf{r}'\in \text{'optically illuminated portion' of } \partial W' \\ \left[\mathbf{0},\mathbf{0}_{\mathrm{i}}\right]_{\mathbf{r}'} & :\mathbf{r}'\in\text{'optically shadowed portion' of }\partial W'\end{cases} \qquad (16)$$

III RESULTS

In Fig. 1 the exact analytical solution for the moving PEC is compared with results relevant to the application of the RMM and of the ROP. As one can see, for such critical geometry (points on the edge of the wedge have non-unique tangential plane) the ROP, that inherits all the limitations of the usual motionless case, is capable to predict only the global bandwidth broadening and the position of the principal in-band peaks, whereas the RMM furnishes a more accurate determination of the whole spectral pattern (in this case a Point-Matching implementation [12] for the RMM was used; the norm of the discretization lattice was taken the same, *mutatis mutandis*, in ROP and RMM for best comparability).

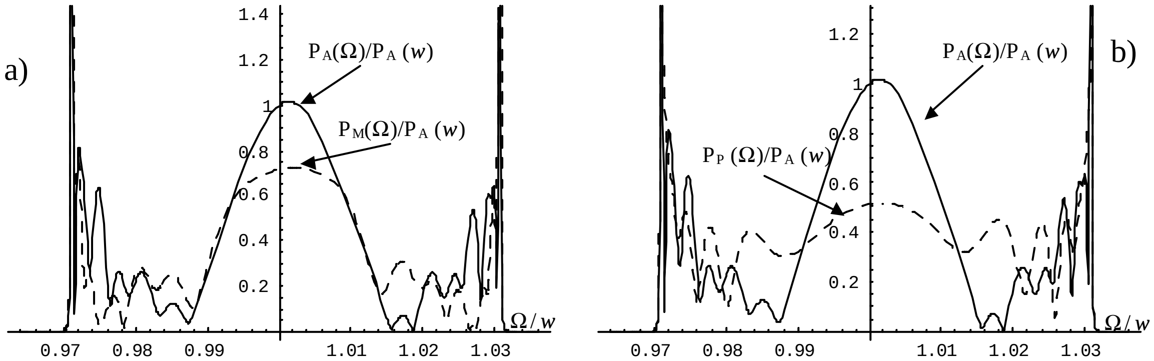

Fig.1 Results relevant to the spectral amplitude of the Poynting vector $\mathrm{P}(\Omega)=\frac{1}{2}\left|\Im_t\left\{\boldsymbol{E}_s(\mathbf{r},t);\Omega\right\}\times\Im_t\left\{\boldsymbol{H}_s(\mathbf{r},t);\Omega\right\}\right|$ vs. normalized frequency Ω/w ($\Im_t\{\ ;\Omega\}$ represent the t®Ω Fourier transform). Continuous line in graphics a) and b): Analytical result [subscript 'A'] for a indefinite wedge: $P_A(\Omega)/P_A(w)$; dotted lines in graphics a) and b): results relevant to application to the case of a finite wedge (height h=25λ, radius r=25λ, where λ=2πc/w) of the RMM [subscript 'M'], $P_M(\Omega)/P_A(w)$, and of the ROP [subscript 'P'], $P_P(\Omega)/P_A(w)$, respectively. Geometric parameters (see [7]): wedge internal angle χ'=5°; y=87.25° is the wedge orientation with respect to the relative velocity 0.02×c; the incident PW propagates along the normal direction, q =270°, with circular polarization; observation point $X=(x,y,z)=(0.9l,0,0)$.

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